Calibration of Chaos Models for Interest Rates

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October 27, 2010

Axiomatic Interest Rate Theory

We recall the following axioms of Hughston and Rafailidis (2005), whereby $(\Omega, \mathcal{F}, P)$ is probability space (physical measure) $\mathcal{F}_t$ is the filtration generated by a (k–dimensional) Brownian motion W_t, S_t are continuous semimartingales and $\xi_t > 0$ is an adapted price process (natural numeraire):

1. There exists a strictly increasing asset with absolutely continuous price process B_t (bank account).
2. If S_t is the price of any asset with an adapted dividend rate D_t then
$$\frac{S_t}{\xi_t} + \int_0^t \frac{D_s}{\xi_s} ds \qquad \text{is a martingale} \tag{1}$$
3. There exists an asset that offers a dividend rate sufficient to ensure that the value of the asset remains constant (floating rate note).
4. There exists a system of discount bond price processes P_{tT} satisfying
$$\lim_{T\to\infty} P_{tT} = 0.$$

The state price density

- Define $V_t = 1/\xi_t$ (state price density).
- Since $B_t V_t$ is a martingale (A2) and B_t is strictly increasing (A1), we have

$$E_t[V_T] = E_t\left[\frac{B_T V_T}{B_T}\right] < E_t\left[\frac{B_T V_T}{B_t}\right] = \frac{B_t V_t}{B_t} = V_t,$$

  which means that V_t is a positive supermartingale.
- Writing $B_t = B_0 \exp\left(\int_0^t r_s ds\right)$ for an adapted process $r_t > 0$ and

$$d(B_t V_t) = -(B_t V_t)\lambda_t dW_t,$$

  for an adapted vector process λ_t, we have that the dynamics for V_t is

$$dV_t = -r_t V_t dt - V_t \lambda_t dW_t. \tag{2}$$

Conditional variance representation

- Integrating (2), taking conditional expectations and the limit $T \to \infty$ (all well–defined thanks to (A3) and (A4)) leads to

$$V_t = E_t \left[\int_t^\infty r_s V_s ds \right].$$

- Now let σ_t be a vector process satisfying $\sigma_t^2 = r_t V_t$ and define the square integrable random variable

$$X_\infty := \int_0^\infty \sigma_s dW_s.$$

- It then follows from the Ito isometry that

$$V_t = E_t \left[(X_\infty - X_t)^2 \right], \tag{3}$$

where $X_t := E_t[X_\infty] = \int_0^t \sigma_s dW_s$.

Wiener chaos

- It is well known that any $X \in L^2(\Omega, \mathcal{F}_\infty, P)$ can be represented as a Wiener chaos expansion

$$X = \sum_{n=0}^{\infty} J_n(\phi_n), \tag{4}$$

where

$$\phi_n \mapsto J_n(\phi_n) = \int_{\Delta_n} \phi_n(s_1, \ldots, s_n) dW_{s_1} \ldots dW_{s_n}. \tag{5}$$

- The deterministic functions $\phi_n \in L^2(\Delta_n)$ are called the chaos coefficients and are uniquely determined by the random variable X.

First order chaos

- In a first order chaos model we have

$$X_\infty = \int_0^\infty \phi(s)dW_s.$$

- In this case $\sigma_s = \phi(s)$, so that $M_{ts} := E_t[\sigma_s^2] = \phi^2(s)$ and

$$V_t = \int_t^\infty M_{ts}ds = \int_t^\infty \phi^2(s)ds$$

- This corresponds to a deterministic interest rate theory, since

$$P_{tT} = \frac{\int_T^\infty \phi^2(s)ds}{\int_t^\infty \phi^2(s)ds}, \quad f_{tT} = \frac{\phi^2(T)}{\int_T^\infty \phi^2(s)ds} = r_T.$$

- The remaining asset prices can be stochastic, however. Indeed, for a derivative with payoff H_T we have

$$H_t = \frac{E_t[V_T H_T]}{V_t} = \frac{V_T}{V_t}E_t[H_T] = P_{tT}E_t[H_T]$$

Factorizable second order chaos: definition

- In a second order chaos model we have

$$X_\infty = \int_0^\infty \phi_1(s)dW_s + \int_0^\infty \int_0^s \phi_2(s,u)dW_u dW_s$$

- This is said to be factorizable when $\phi_1(s) = \alpha(s)$ and $\phi_2(s,u) = \beta(s)\gamma(u)$.
- In this case, $\sigma_s = \phi(s) + \beta(s)R_s$ where

$$R_t = \int_0^t \gamma(s)dW_s$$

  is a martingale with quadratic variation $Q(t) = \int_0^t \gamma^2(s)ds$.
- Notice that the scalar random variable R_t is the sole state variable for the interest rate model at time t, even in the case of a multidimensional Brownian motion W_t.

Factorizable second order chaos: bond prices

- Defining $Z_{tT} = \int_T^\infty M_{ts} ds$, we see that bond prices are given by

$$P_{tT} = \frac{Z_{tT}}{Z_{tt}}.$$

- Integrating the expression for M_{ts} gives

$$Z_{tT} = \int_T^\infty M_{ts} ds = A(T) + B(T)R_t + C(T)(R_t^2 - Q(t)),$$

where

$$\begin{aligned} A(T) &= \int_T^\infty (\alpha^2(s) + \beta^2(s)Q(s))ds \\ B(T) &= 2\int_T^\infty \alpha(s)\beta(s)ds, \quad C(T) = \int_T^\infty \beta^2(s)ds \end{aligned}$$

- Therefore

$$P_{tT} = \frac{A(T) + B(T)R_t + C(T)(R_t^2 - Q(t))}{A(t) + B(t)R_t + C(t)(R_t^2 - Q(t))}$$

Factorizable second order chaos: option prices

- The price at time zero of an option with payoff $(P_{tT} - K)^+$ is

$$c(0,t,T,K) = \frac{1}{V_0} E\left[V_t\left(P_{tT} - K\right)^+\right] = \frac{1}{V_0} E\left[\left(Z_{tT} - KZ_{tt}\right)^+\right],$$

- Fixing t, T and K, it follows that

$$Z_{tT} - KZ_{tt} = A + BY + CY^2,$$

where $Y = R(t)/\sqrt{Q(t)} \sim N(0,1)$ and

$$\begin{aligned} A &= [A(T) - KA(t)] - [C(T) - KC(t)]Q(t) \\ B &= [B(T) - KB(t)]\sqrt{Q(t)}, \quad C = [C(T) - KC(t)]Q(t) \end{aligned}$$

- Therefore, defining $p(y) = A + By + Cy^2$, we have

$$c(0,t,T,K) = \frac{1}{A(0)\sqrt{2\pi}} \int_{p(y)\geq 0} p(y) e^{-\frac{1}{2}y^2} dy,$$

which can be calculated explicitly in terms of the roots of the polynomial $p(y)$.

- Analogous expressions can be derived for puts, swaptions, caps, floors, etc...

One–variable third order chaos

- Consider now

$$X_\infty = \int_0^\infty \alpha(s)dW_s + \int_0^\infty\int_0^s \beta(s)dW_u dW_s + \int_0^\infty\int_0^s\int_0^u \delta(s)dW_v dW_u dW_s$$
$$= \int_0^\infty \left[\alpha(s) + \beta(s)W_s + \frac{1}{2}\delta(s)(W_s^2 - s)\right] dW_s$$

- For fitting the initial term structure P_{0T}, this behaves like a first order chaos with $\phi(s) = \alpha^2(s) + \beta^2(s)s + \delta^2(s)s^2/2$.
- Moreover, since

$$Z_{tT} = a(T) + b(T)W_t + c(T)W_t^2 + d(T)W_t^3 + e(T)W_t^4,$$

general bond prices are expressed as the ratio of 4th–order polynomials in W_t.

- Similarly, option prices can be found explicitly by integrating a 4th–order polynomial of a standard normal random variable.

Data

- For P_{0T} we use clean prices from the UK Debt Management Office (DMO) at 146 dates (every other business day) from January 1998 to January 1999 with 50 maturities for each date.
- We also use weekly data at 157 dates (every Friday) from December 2002 to December 2005 with about 120 maturities for each date.
- For joint calibration with option prices we also consider yield data from money market at 53 dates (every Friday) from September 2000 to August 2001 with 17 maturities for each date, together with ATM caps (37 caplets) and swaptions (6 maturities and 7 tenors).

Parametric specification

- Motivated by the vast literature on forward rate curve fitting (so-called descriptive–form interest rate models), we consider the exponential–polynomial family (Bjork and Christensen 99):

$$\phi(s) = \sum_{i=1}^{n} \left(\sum_{j=1}^{\mu_i} b_{ij} s^j \right) e^{-c_i s}$$

- Special cases in this family are the Nelson–Sigel (87), Svensson (94) and Cairns (98) models:

$$\begin{aligned}
\phi_{NS}(s) &= b_0 + (b_1 + b_2 s) e^{-c_1 s} \\
\phi_{Sv}(s) &= b_0 + (b_1 + b_2 s) e^{-c_1 s} + b_3 s e^{-c_2 s} \\
\phi_C(s) &= \sum_{i=1}^{4} b_1 e^{-c_i s}
\end{aligned}$$

Descriptive fit for yield curves

Chaos fit for yield curves

Calibration results: bonds from Jan/98 to Feb/99

	Model	N	-L	RMSPE (%)	DM
Sv	Svensson	6	160	0.70	-
NS	Nelson–Siegel	4	2101	2.67	-4.45
1	1st chaos	3	4420	4.44	-11.46
2	1st chaos	5	250	0.86	-3.54
3	one-var 2nd chaos	6	162	0.82	-2.26
4	one-var 2nd chaos	7	160	0.69	0.22
5	one-var 2nd chaos	7	145	0.75	-1.05
6	factorizable 2nd chaos	6	335	0.88	-2.54
7	factorizable 2nd chaos	6	245	0.68	0.27
8	factorizable 2nd chaos	6	1245	1.26	-3.81
9	factorizable 2nd chaos	7	179	0.63	1.38
10	factorizable 2nd chaos	7	153	0.72	-1.07
11	one-var 3rd chaos	6	168	0.72	-1.24
12	one-var 3rd chaos	7	141	0.76	-1.16
13	one-var 3rd chaos	7	152	0.72	-1.19
14	one-var 3rd chaos	7	149	0.76	-1.43

Calibration results: bonds from Dec/01 to Dec/05

	Model	N	-L	RMSPE (%)	DM
Sv	Svensson	6	442	0.76	-
NS	Nelson–Siegel	4	541	0.97	-1.76
1	1st chaos	3	8716	3.96	-3.50
2	1st chaos	5	438	0.99	-1.99
3	one-var 2nd chaos	6	388	0.89	-1.23
4	one-var 2nd chaos	7	388	0.80	-0.38
5	one-var 2nd chaos	7	329	0.66	1.26
6	factorizable 2nd chaos	6	437	1.04	-3.33
7	factorizable 2nd chaos	6	495	0.84	-0.68
8	factorizable 2nd chaos	6	421	1.19	-2.84
9	factorizable 2nd chaos	7	365	0.82	-0.78
10	factorizable 2nd chaos	7	323	0.72	0.36
11	one-var 3rd chaos	6	388	0.87	-1.06
12	one-var 3rd chaos	7	350	0.78	-0.11
13	one-var 3rd chaos	7	367	0.68	1.24
14	one-var 3rd chaos	7	325	0.69	0.60

Forward rates

Models for option price calibration

We consider the following models for option price calibration:

- The Hull–White model with Svensson term structure (8 parameters):

$$dr_t = \kappa(\Theta(t) - r_t) + \sigma\sqrt{r_t}dW_t$$
$$f_{0t} = b_0 + (b_1 + b_2 t)e^{-c_1 t} + b_3 t e^{-c_2 t}$$

- The rational lognormal model with Nakamura-Yu parametrization and Svensson term structure (9 parameters):

$$P_{tT} = \frac{G_1(T)M_t + G_2(T)}{G_1(t)M_t + G_2(t)}$$
$$G_1(t) = \frac{\alpha}{\gamma+1}(P_{0t})^{\gamma+1}, G_2(t) = P_{0t} - G_1(t), \quad M_t = e^{\beta W_t - \frac{1}{2}\beta^2 t}$$
$$f_{0t} = b_0 + (b_1 + b_2 t)e^{-c_1 t} + b_3 t e^{-c_2 t}$$

- The lognormal forward LIBOR model with Rebonato volatility, Schoenmakers and Coffey correlation and Svensson term structure (13 parameters):

$$dF_t^j = \sigma_j(t) F_t^j dZ_t^j$$

Hull–White fit for yields and caplets

Rational lognormal fit for yields and caplets

LFM fit for yields and caplets

Chaos fit for yields and caplets

Hull–White fit for yields and swaptions

Rational lognormal fit for yields and swaptions

LFM fit for yields and swaptions

Chaos fit for yields and swaptions

ATM caplet calibration results

Table: Yield and ATM caplet calibration for 2000-2001

No.	Model	N	TotalE1	YieldE	CplE	SwpE
1	one-var 2nd chaos	6	5.1	2.0	4.6	14.9
2	one-var 2nd chaos	7	3.3	1.7	2.7	16.3
3	factorizable 2nd	6	3.8	2.1	3.1	26.5
4	one-var 3rd chaos	6	4.2	2.0	3.5	15.5
5	one-var 3rd chaos	7	3.2	1.3	2.9	15.7
6	one-var 3rd chaos	9	2.6	1.1	2.3	17.0
I	Hull-White	8	8.7	0.6	8.7	25.8
II	Rational-log	9	9.2	0.6	9.2	13.9
III	LFM	10	3.0	0.6	3.0	-

ATM swaption calibration results

Table: Yield and ATM swaption calibration for 2000 – 2001

No.	Model	N	TotalE2	YieldE	SwpE	CplE
1	one-var 2nd chaos	6	7.1	1.8	6.8	14.5
2	one-var 2nd chaos	7	7.1	2.0	6.7	14.6
3	factorizable 2nd	6	7.1	2.1	6.8	14.3
4	one-var 3rd chaos	6	5.3	2.9	4.1	10.2
5	one-var 3rd chaos	7	3.8	1.5	3.4	8.6
6	one-var 3rd chaos	9	3.5	1.5	3.1	9.1
I	Hull-White	8	10.2	0.6	10.2	17.6
II	Rational-log	9	8.4	0.6	8.4	15.3
III	LFM	13	5.0	0.6	5.0	8.1

Joint calibration results

Table: Yield, ATM caplet and ATM swaption calibration for $2000 - 2001$

No.	Model	N	TotalE3	YieldE	SwpE	CplE
1	one-var 2nd chaos	6	12.5	2.2	9.3	7.9
2	one-var 2nd chaos	7	12.1	2.4	9.3	7.3
3	factorizable 2nd	6	12.1	2.6	8.4	8.2
4	one-var 3rd chaos	6	8.2	4.3	4.4	5.2
5	one-var 3rd chaos	7	7.1	1.6	4.4	5.2
6	one-var 3rd chaos	9	5.9	2.2	4.1	3.4
I	Hull-White	8	18.4	0.6	12.2	13.7
II	Rational-log	9	14.6	0.6	10.0	10.6
III	LFM	13	6.5	0.6	5.5	3.1

Model selection

Table: AIC model selection relative frequency (first dataset)

Model	Cpl	SW	JT
One-var 3rd, 7 par	$\frac{2}{53}$	$\frac{50}{53}$	$\frac{23}{53}$
LFM	$\frac{51}{53}$	$\frac{3}{53}$	$\frac{30}{53}$

Model	Cpl	SW	JT
One-var 3rd, 9 par	$\frac{36}{53}$	$\frac{53}{53}$	$\frac{39}{53}$
LFM	$\frac{17}{53}$	$\frac{0}{53}$	$\frac{14}{53}$

Table: AIC model selection relative frequency (second dataset)

Model	Cpl	SW	JT
One-var 3rd, 7 par	$\frac{14}{53}$	$\frac{23}{53}$	$\frac{7}{53}$
LFM	$\frac{39}{53}$	$\frac{30}{53}$	$\frac{46}{53}$

Model	Cpl	SW	JT
One-var 3rd, 9 par	$\frac{52}{53}$	$\frac{44}{53}$	$\frac{39}{53}$

Conclusions

1. We propose a systematic way to calibrate interest rate model in the chaotic approach.
2. For term structure calibration, 3rd chaos models perform comparably to the Svensson model, with the advantage of being fully stochastic and consistent with non-arbitrage and positivity conditions.
3. For ATM option calibration, a 3rd chaos model with 9 parameters outperforms the lognormal forward LIBOR models.
4. Further work will compare chaos and SABR for joint smile calibration (caplets and swaptions).