Wiener chaos and the Cox–Ingersoll–Ross model

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1. Introduction

- We take the CIR model as specified by the short rate process

$$dr_t = a(b - r_t)dt + c\sqrt{r_t}d\widetilde{W}_t, \tag{1}$$

  for some positive constants a, b, c with $4ab > c^2$, where $\widetilde{W}_t$ is a standard one dimensional Brownian motion under the "physical" measure P, and by a market price of risk λ_t taken to be proportional to $\sqrt{r}$.

- We seek for the chaotic representation of the underlying random variable X_∞ in the Hughston/Rafailidis framework.

2. Positive Interest Rates

2.1 State price density and the potential approach

Let $P_{tT}, 0 \leq t \leq T$ denote the price at time t for a zero coupon bond which pays one unit of currency at its maturity T. Clearly $P_{tt} = 1$ for all $0 \leq t < \infty$ and furthermore, positivity of the interest rate is equivalent to having

$$P_{ts} \leq P_{tu}, \tag{2}$$

for all $0 \leq t \leq u \leq s$.

A general way to model bond prices [Rogers 97] is to write

$$P_{tT} = \frac{E_t[V_T]}{V_t}, \tag{3}$$

for a positive adapted continuous process V_t, called the *state price density*.

Positivity of the interest rates is then equivalent to V_t being a supermartingale. To match the initial term structure, we must have $E[V_T] = P_{0T}$. If we further impose that $P_{0T} \to 0$ as $T \to \infty$, then V_t satisfies all the properties of a potential. This can then be uniquely expressed as

$$V_t = E_t[A_\infty] - A_t, \tag{4}$$

for an increasing process A_t satisfying the constraint

$$E\left[\frac{\partial A_T}{\partial T}\right] = -\frac{\partial P_{0T}}{\partial T}. \tag{5}$$

2.2 Related quantities

Flesaker and Hughston [96] observed that any arbitrage free system of zero coupon bond prices has the form

$$P_{tT} = \frac{\int_T^\infty h_s M_{ts} ds}{\int_t^\infty h_s M_{ts} ds}, \qquad \text{for } 0 \le t \le T < \infty. \tag{6}$$

Here $h_T = -\frac{\partial P_{0T}}{\partial T}$ is a positive deterministic function obtained from the initial term structure and M_{ts} is a family of strictly positive continuous martingales satisfying $M_{0s} = 1$. Any such system of prices can be put into a potential form by setting

$$V_t = \int_t^\infty h_s M_{ts} ds. \tag{7}$$

The converse result is less direct and was first established by Jin and Glasserman [01].

These equivalent ways of modelling positive interest rates can now be related to other standard financial objects: given a positive supermartingale V_t, there exists a unique positive (local) martingale Λ_t such that the process $B_t = \Lambda_t / V_t$ is strictly increasing and $V_0 = \Lambda_0$. We identify B_t with a riskless money market account initialized at $B_0 = 1$ and write it as

$$B_t = \exp(\int_0^t r_s ds), \tag{8}$$

for an adapted process $r_s > 0$, the short rate process.

The market price of risk then arises as the adapted vector valued process λ_t such that

$$d\Lambda_t = -\Lambda_t \lambda_t^\dagger dW_t, \qquad \Lambda_0 = 1, \tag{9}$$

where W is an N–dimensional P–Brownian motion, from what it follows that

$$dV_t = -r_t V_t dt - V_t \lambda_t^\dagger dW_t, \qquad V_0 = 1. \tag{10}$$

so that the specification of the process V_t is enough to produce both the short rate r_t and the market price of risk λ_t.

2.3 The Chaotic Approach

Assume that the state price density V_t is a potential satisfying

$$E\left[\int_0^\infty r_s V_s ds\right] < \infty \tag{11}$$

By integrating (10) on the interval (t, T), taking conditional expectations at time t and the limit $T \to \infty$, one finds that

$$V_t = E_t\left[\int_t^\infty r_s V_s ds\right] \tag{12}$$

Now let σ_t be a vector valued process such that

$$\|\sigma_t\|^2 = r_t V_t, \tag{13}$$

and define the square integrable random variable

$$X_\infty = \int_0^\infty \sigma_s dW_s. \tag{14}$$

It follows from the Ito isometry that

$$V_t = E_t[X_\infty^2] - E_t[X_\infty]^2, \tag{15}$$

which is called the conditional variance representation of the state price density V_t. A direct comparison between (12) and (7) gives that

$$h_s M_{ts} = E_t\left[\|\sigma_s\|^2\right]. \tag{16}$$

Similarly, by comparing the conditional variance representation (15) with the decomposition (4), we see that

$$E_t[X_\infty^2] - X_t^2 = E_t[A_\infty] - A_t,$$

where $X_t = E_t[X_\infty]$. It follows from the uniqueness of the Doob-Meyer decomposition that

$$A_t = [X, X]_t,$$

that is, the quadratic variation of the process X_t.

2.4 Wiener chaos

Let W_t be an N−dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, P)$. We introduce a compact notation

$$\tau = (s, \mu) \in \Delta \doteq \mathbb{R}_+ \times \{1, \ldots, N\}$$

and express integrals as

$$\begin{aligned} \int_\Delta f(\tau) d\tau &\doteq \sum_\mu \int_0^\infty f(s,\mu) ds, \\ \int_\Delta f(\tau) dW_\tau &\doteq \sum_\mu \int_0^\infty f(s,\mu) dW_s^\mu. \end{aligned} \tag{17}$$

For each $n \geq 0$, let

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \tag{18}$$

be the nth Hermite polynomial. For $h \in L^2(\Delta)$, let $\|h\|^2 = \int_\Delta h(\tau)^2 d\tau$ and define the Gaussian random variable

$$W(h) := \int_\Delta h(\tau) dW_\tau.$$

The spaces

$$\begin{aligned} \mathcal{H}_n &\doteq \text{span}\{H_n(W(h)) | h \in L^2(\Delta)\}, \quad n \geq 1, \\ \mathcal{H}_0 &\doteq \mathbb{C}, \end{aligned}$$

form an orthogonal decomposition of the space $L^2(\Omega, \mathcal{F}_\infty, P)$ of square integrable random variables:

$$L^2(\Omega, \mathcal{F}_\infty, P) = \oplus_{n=0}^{\infty} \mathcal{H}_n.$$

Each $\mathcal{H}_n$ can be identified with $L^2(\Delta_n)$ via the isometries

$$J_n : L^2(\Delta_n) \to \mathcal{H}_n$$

given by

$$f_n \mapsto J_n(f_n) = \int_{\Delta_n} f_n(\tau_1, \ldots, \tau_n) dW_{\tau_1} \ldots dW_{\tau_n}, \tag{19}$$

where $\Delta_n \doteq \{(\tau_1, \ldots, \tau_n) | \tau_i = (s_i, \mu_i) \in \Delta, 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n < \infty\}$.

With these ingredients, one is then led to the result that any $X \in L^2(\Omega, \mathcal{F}_\infty, P)$ can be represented as a *Wiener chaos expansion*

$$X = \sum_{n=0}^{\infty} J_n(f_n), \tag{20}$$

where the deterministic functions $f_n \in L^2(\Delta_n)$ are uniquely determined by the random variable X.

A special example arises by noting that for $h \in L^2(\Delta)$

$$n! J_n(h^{\otimes n}) = \|h\|^n H_n\left(\frac{W(h)}{\|h\|}\right) \tag{21}$$

and furthermore

$$\exp\left[W(h) - \frac{1}{2}\int |h(\tau)|^2 d\tau\right] = \sum_{n=0}^{\infty} \frac{\|h\|^n}{n!} H_n\left(\frac{W(h)}{\|h\|}\right) \tag{22}$$

In the notation of quantum field theory, this example defines the Wick ordered exponential and Wick powers

$$\begin{aligned} :\exp[W(h)]: \quad &\doteq \quad \exp\left[W(h) - \frac{1}{2}\int |h(\tau)|^2 d\tau\right] \\ :W(h)^n: \quad &\doteq \quad n! J_n(h^{\otimes n}) \end{aligned} \tag{23}$$

Theorem 1 *For any random variable $X \in L^2(\Omega, \mathcal{F}_\infty, P)$, the generating functional $Z_X(h) : L^2(\Delta) \to \mathbb{C}$ defined by*

$$Z_X(h) \doteq E\left[X \exp\left[W(h) - \frac{1}{2}\int |h(\tau)|^2 d\tau\right]\right] \tag{24}$$

is an entire analytic functional of $h \in L^2(\Delta)$ and hence has an absolutely convergent expansion

$$Z_X(h) = \sum_{n\geq 0} F_X^{(n)}(h) \tag{25}$$

where

$$F_X^{(n)}(h) = \int_{\Delta_n} f_X^{(n)}(\tau_1, \ldots, \tau_n) h(\tau_1) \ldots h(\tau_n) d\tau_1 \ldots d\tau_n \tag{26}$$

Here, $f_X^{(n)}(\tau_1, \ldots, \tau_n)$ is the nth Frechet derivative of Z_X at $h = 0$. Finally, the Wiener–Itô chaos expansion of X is

$$X = \sum_{n\geq 0} \int_{\Delta_n} f_X^{(n)}(\tau_1, \ldots, \tau_n) dW_{\tau_1} \ldots dW_{\tau_n} \tag{27}$$

3. Squared Gaussian models

The CIR model with an integer constraint $N \doteq \frac{4ab}{c^2} \in \mathbb{N}_+ \setminus \{0, 1\}$ lies in the class of so–called squared Gaussian models. By introducing an $\mathbb{R}^N$–valued Ornstein–Uhlenbeck process R_t, governed by the stochastic differential equation

$$dR_t = -\frac{a}{2} R_t \; dt + \frac{c}{2} dW_t \tag{28}$$

where W_t is N–dimensional Brownian motion, one verifies that the square $r_t = \|R_t\|^2$ satisfies (1) where

$$\widetilde{W}_t = \int_0^t \|R_t\|^{-1} R_t \cdot dW_t$$

is itself a one–dimensional Brownian motion.

Definition 2 *A pair* (r_t, λ_t) *of* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *processes is called an* $N-$*dimensional squared Gaussian model of interest rates (*$N \geq 2$*) if there is an* $\mathbb{R}^N-$*valued Ornstein–Uhlenbeck process such that* $r_t = R_t^\dagger R_t$ *and* $\lambda_t = \bar{\lambda}(t) R_t$*.* R_t *satisfies*

$$dR_t = \alpha(t)(\bar{R}(t) - R_t)dt + \gamma(t)dW_t, \quad R|_{t=0} = R_0, \tag{29}$$

where $\alpha, \gamma, \bar{\lambda}$ *are symmetric matrix valued and* $\bar{R}$ *vector valued deterministic Lipschitz functions on* $\mathbb{R}_+$ *and* W *is standard* $N-$*dimensional Brownian motion. In addition we impose boundedness conditions that there is some constant* $M > 0$ *such that*

$$\begin{aligned} \bar{\lambda}^2(t) &\leq MI, \quad \alpha(t) \geq M^{-1}I, \\ \alpha(t) + \gamma(t)\bar{\lambda}(t) &\geq M^{-1}I, \quad \gamma^2(t) \geq M^{-1}I, \end{aligned} \tag{30}$$

for all t*.*

The exact solution of (29) is easily seen to be

$$R_t = \tilde{R}(t) + \int K(t, t_1)(\gamma dW)_{t_1} \tag{31}$$

where

$$\tilde{R}(t) = K(t, 0)R_0 + \int K(t, t_1)\alpha(t_1)\bar{R}(t_1)dt_1 \tag{32}$$

and $K(t, s), t \geq s$ is the matrix valued solution of

$$\begin{cases} dK(t, s)/dt = -\alpha(t)K(t, s) & 0 \leq s \leq t \\ K(t, t) = I & 0 \leq t \end{cases} \tag{33}$$

which generates the Ornstein–Uhlenbeck semigroup.

In accordance with (10), we define the state price density process is

$$V_t = \exp\left[-\int_0^t \left(R_s^\dagger \left(1 + \frac{\bar{\lambda}^2}{2}\right) R_s ds + R_s^\dagger \bar{\lambda} dW_s\right)\right] \tag{34}$$

We thus have a natural candidate for the random variable X_∞:

$$X_\infty = \int_0^\infty \sigma_t^\dagger \, dW_t \tag{35}$$

where the $\mathbb{R}^N$–valued process

$$\sigma_t \doteq \exp\left[-\int_0^t \left(R_s^\dagger \left(\frac{1}{2} + \frac{\bar{\lambda}^2}{4}\right) R_s ds + \frac{1}{2} R_s^\dagger \bar{\lambda} dW_s\right)\right] R_t \tag{36}$$

is the natural solution of $\sigma_t^\dagger \sigma_t = r_t V_t$. We can then prove that Λ_t is a martingale for $0 \leq t \leq T$ and the state price density V_t is a potential.

4. Exponentiated second chaos

The chaos expansion we seek for the CIR model will be derived from a closed formula for expectations of e^{-Y} for elements

$$Y = A + \int_{\Delta} B(\tau_1) dW_{\tau_1} + \int_{\Delta_2} C(\tau_1, \tau_2) dW_{\tau_1} dW_{\tau_2} \tag{37}$$

in a certain subset $\mathcal{C}^+ \subset \mathcal{H}_{\leq 2} \doteq \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$.

If in (37) we define $C(\tau_1, \tau_2) = C(\tau_2, \tau_1)$ when $\tau_1 > \tau_2$, then C is the kernel of a symmetric integral operator on $L^2(\Delta)$:

$$[Cf](\tau) = \int_0^\infty C(\tau, \tau_1) f(\tau_1) d\tau_1. \tag{38}$$

We say that $Y \in \mathcal{H}_{\leq 2}$ is in $\mathcal{C}^+$ if C is the kernel of a symmetric Hilbert–Schmidt operator on $L^2(\Delta)$ such that $(1 + C)$ has non–negative spectrum.

Proposition 3 *Let $Y \in \mathcal{C}^+$. Then*

$$E[e^{-Y}] = [\det_2(1+C)]^{-1/2} \exp\left[-A + \frac{1}{2}\int_{\Delta_2} B(\tau_1)(1+C)^{-1}(\tau_1,\tau_2)B(\tau_2)d\tau_1 d\tau_2\right].$$

Remark 4 *The Carleman–Fredholm determinant is defined as the extension of the formula*

$$\det_2(1+C) = \det(1+C)\exp[-\mathrm{Tr}(C)] \tag{39}$$

from finite rank operators to bounded Hilbert–Schmidt operators; the operator kernel $(1+C)^{-1}(\tau_1,\tau_2)$ is also the natural extension from the finite rank case.

Corollary 5 (Wick's theorem) *The random variable* $X = e^{-Y}$*, for* $Y = \int_{\Delta_2} C(\tau_1, \tau_2) dW_{\tau_1} dW_{\tau_2} \in \mathcal{C}^+$ *has Wiener chaos coefficient functions*

$$f_n(\tau_1, \ldots, \tau_n) = \begin{cases} K \sum_{G \in \mathcal{G}_n} \prod_{g \in G} [C(1+C)^{-1}](\tau_{g_1}, \tau_{g_2}) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

where $K = [\det_2(1 + \mathrm{C})]^{-1/2}$ *and for* n *even,* $\mathcal{G}_n$ *is the set of* Feynman graphs *on the* n *marked points* $\{\tau_1, \ldots, \tau_n\}$*. Each Feynman graph* G *is a disjoint union of unordered pairs* $g = (\tau_{g_1}, \tau_{g_2})$ *with* $\cup_{g \in G}\, g = \{\tau_1, \ldots, \tau_n\}$*.*

5. **The chaotic expansion for squared Gaussian models**

We now derive the chaos expansion for the squared Gaussian model defined by (29). In view of (35) it will be enough to find the chaos expansion for σ_T^μ, $T < \infty$. We start by finding its the generating functional $Z_{\sigma_T^\mu}$. Define the auxiliary functional $Z(h,k) = E\left[e^{-Y_T}\right]$ with

$$\begin{aligned} Y_T \;&=\; \int_0^T R_t^\dagger \left(\frac{I}{2} + \frac{\bar{\lambda}^2}{4}\right) R_t dt + \frac{1}{2}\int_0^T R_t^\dagger \bar{\lambda}(t) dW_t - \int_0^T h^\dagger(t) dW_t \\ &\quad + \frac{1}{2}\int_0^T h^\dagger(t) h(t) dt - \int_0^T k^\dagger(t) R_t dt. \qquad (40) \end{aligned}$$

We want to use Proposition 3 in order to comnpute $Z(h,k)$. Substitution of (31) puts the exponent Y_T in the form of (37) with

$$
\begin{aligned}
A_T &= \int_0^T \left[\tilde{R}^\dagger(t)\left(\frac{I}{2}+\frac{\bar{\lambda}(t)^2}{4}\right)\tilde{R}(t)+\frac{1}{2}h^\dagger(t)h(t)-k^\dagger(t)\tilde{R}(t)\right]dt \\
&+\int_0^T \mathrm{tr}\left\{\gamma(t)\left[\int_0^T K_T^\dagger(t,s)\left(\frac{I}{2}+\frac{\bar{\lambda}(t)^2}{4}\right)K_T(s,t)ds\right]\gamma(t)\right\}dt \\
&\frac{1}{2}\int_0^T \mathrm{tr}\left[\int_0^T \gamma(s)K_T^\dagger(s,t)\bar{\lambda}(t)ds\right]dt, \\
B_T(t) &= -h(t)-\gamma(t)\int_0^T K_T^\dagger(t,s)k(s)ds+\frac{1}{2}\bar{\lambda}(t)\tilde{R}(t) \\
&+\gamma(t)\int_0^T K_T^\dagger(t,s)\left(I+\frac{\bar{\lambda}(t)^2}{2}\right)\tilde{R}(s)ds, \\
C_T(t_1,t_2) &= \gamma(t_1)\left[\int_0^T K_T^\dagger(t_1,s)\left(I+\frac{\bar{\lambda}(t)^2}{2}\right)K_T(s,t_2)ds\right]\gamma(t_2) \\
&+\frac{1}{2}\left[\gamma(t_1)K_T^\dagger(t_1,t_2)\bar{\lambda}(t)+\bar{\lambda}(t)K_T(t_1,t_2)\gamma(t_2)\right].
\end{aligned}
$$

Therefore, we can use Proposition 3 for $E[e^{-Y_T}]$, leading to a general formula for the generating functional $Z(h,k)$. Differentiation once with respect to k then yields an expression for $Z_{\sigma_T}(h)$.

These formulas simplify considerably if the function $\tilde{R}$ vanishes, which is true in the simple CIR model of (28) when $r_0 = 0$. In this case we have $\alpha(t) = \frac{a}{2}I$ and $\gamma(t) = \frac{c}{2}I$, so that $K_T(s,t) = e^{-a(s-t)/2}\mathbb{1}(t \le s \le T)$ and

$$C_T(t_1,t_2) \;=\; \frac{c^2}{4a}\left(I + \frac{\bar{\lambda}^2}{2}\right)\left[e^{-\frac{a}{2}|t_1-t_2|} - e^{\frac{a}{2}(t_1+t_2-2T)}\right] + \frac{c}{2}\bar{\lambda}e^{-\frac{a}{2}|t_1-t_2|}.$$

The previous expression for $Z_{\sigma_T}(h)$ reduces to

$$\begin{aligned} Z_{\sigma_T}(h) &= M_T \left[K_T \gamma (1 + C_T)^{-1} h \right] (T) \\ & \exp \left[-\frac{1}{2} \int_0^T h_t^\dagger h_t dt + \frac{1}{2} \int_{\Delta_2} h_{t_1}^\dagger (1 + C_T)^{-1}(t_1, t_2) h_{t_2} dt_1 dt_2 \right], \end{aligned}$$

where

$$M_T = e^{-\frac{1}{2}\mathrm{tr} C_T} (\det{}_2(1 + C_T))^{-1/2} = (\det(1 + C_T))^{-1/2}. \quad (41)$$

Theorem 6 *The nth term of the chaos expansion of σ_T for the CIR model with initial condition $r_0 = 0$ is zero for n even. For n odd, the kernel of the expansion is the function $f_T^{(n)}(\cdot) : \Delta_n \to \mathbb{R}$*

$$f_T(t_1, \dots, t_n) = M_T \sum_{G \in \mathcal{G}_n^*} \prod_{g \in G} L(g), \tag{42}$$

where

$$L(g) = \begin{cases} [C_T(1 + C_T)^{-1}](t_{g_1}, t_{g_2}) & T \notin g \\ \\ (K_T \gamma (1 + C_T)^{-1})(T, t_{g_2}) & T \in g. \end{cases} \tag{43}$$

Here, $\mathcal{G}_n^*$ is the set of Feynman graphs, *each Feynman graph G being a partition of $\{t_1, \dots, t_n, T\}$ into pairs $g = (t_{g_1}, t_{g_2})$.*

The chaos expansion for X_∞ itself is exactly the same, except that the variable T is treated as an additional Itô integration variable. The explicit expansion up to fourth order is:

$$
\begin{aligned}
X_\infty \;=\; & \int_{\Delta_2} M_T[K_T\gamma(1+C_T)^{-1}](T,t_1)dW_{t_1}dW_T \\
& + \int_{\Delta_4} M_T[K_T\gamma(1+C_T)^{-1}](T,t_3)[C_T(1+C_T)^{-1}](t_1,t_2)dW_{t_1}dW_{t_2}dW \\
& + \int_{\Delta_4} M_T[K_T\gamma(1+C_T)^{-1}](T,t_2)[C_T(1+C_T)^{-1}](t_1,t_3)dW_{t_1}dW_{t_2}dW \\
& + \int_{\Delta_4} M_T[K_T\gamma(1+C_T)^{-1}](T,t_1)[C_T(1+C_T)^{-1}](t_2,t_3)dW_{t_1}dW_{t_2}dW \\
& \cdots
\end{aligned}
$$

6. Discussion

- The CIR model, at least in integer dimensions, can be viewed within the chaos framework of Hughston and Rafailidis as arising from a random variable X_∞ derived from exponentiated second chaos random variables $e^{-Y}, Y \in \mathcal{C}^+$. Such exponentiated $\mathcal{C}^+$ variables form a rich and natural family which is likely to include many more candidates for applicable interest rate models.

- Although their analytic properties are complicated, there do exist approximation schemes which can in principle be the basis for numerical methods