A Monte Carlo method for exponential hedging in semimartingale markets

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1. Introduction

Utility methods were made central in economic theories through the works of

- Léon Walras (1874): utility based theory of price (derive demand from utility) and

- Francis Edgeworth (1881): indifference curve.

The concept had been introduced much earlier by

- Daniel Bernoulli (1738): diminishing marginal utility.

The mathematical axiomatization was performed by

- John von Neumann and Oskar Morgenstern (1944) and
- Paul Samuelson (1938/1947).

It culminated with axiomatic equilibrium theory in the work of

- Kenneth Arrow and Gerard Debreu (1959).

In mathematical finance,

- Robert Merton (1969/1971): formulated the problem of investing in a financial market in a way which maximizes the expected utility of the terminal value of the investment. Solved the problem for a complete market with Markovian state process using the methods of stochastic control (Hamilton–Jacobi–Belmann equation);

- Stanley Pliska (1986) (and others): introduced martingale techniques to solve the problem for complete markets but avoiding the Markovian assumption. Obtained the relation

$$\frac{dQ}{dP} = \lambda U'(x^*) \tag{1}$$

  therefore establishing an equilibrium economy argument for the whole subject;

- Karatzas et al (1990): applied duality techniques (Legendre transforms) to solve the problem for (special cases) of incomplete markets.

“ It is sometimes claimed in the economic literature that discussions of the notions of utility and preferences are altogether unnecessary, since these are purely verbal definitions with no empirically observable consequences, i.e., entirely tautological. It does not seem to us that these are qualitatively inferior to certain well established and indispensable notions in physics, like force, mass, charge, etc. That is, while they are in their immediate form merely definitions, they become subject to empirical control through the theories which are built on them - and in no other way. Thus the notion of utility is raised above the status of a tautology by such economic theories as make use of it and the results of which can be compared with experience or at least with common sense. ”

– John von Neumann and Oskar Morgenstern.

2. Utility based hedging for semimartingale markets

The hedging problem is the problem of a market agent who faces a liability B at a time T and must invest in the market over the period $[0, T]$ in a rational way to reduce the risk of the liability.

- Randomness: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$
- Traded assets: $\mathbb{R}^d$–valued *càdlàg* semimartingale $S_t = (S_t^1, \ldots, S_t^d)$
- Liability: $\mathcal{F}_T$–measurable random variable B
- Portfolio: $\mathbb{R}^d$-valued process $H_t = (H_t^1, \ldots, H_t^d) \in L(S)$

For self–financing portfolios, the (discounted) wealth at time t is given by the process

$$X_t = x + (H \cdot S)_t := x + \int_0^t H_u dS_u, \qquad t \in [0, T],$$

where $x \in \mathbb{R}$ is some deterministic initial wealth.

Definition 1 *The class $\mathcal{H}$ of admissible portfolios consists of the process $H \in L(S)$ for which $(H \cdot S)_t$ is P–a.s. uniformly bounded from below.*

Definition 2 *A probability measure Q is called an absolutely continuous (resp. equivalent) local martingale measure for S if $Q \ll P$ (resp. $Q \sim P$) and S is a local martingale under Q.*

Denote by $\mathcal{M}^a(S)$ (resp. $\mathcal{M}^e(S)$) the set of absolutely continuous (resp. equivalent) local martingale measures for S.

Theorem 3 (FTAP,DS/94) *If S is a locally bounded semimartingale, then there exists an equivalent local martingale measure Q for S if and only if S satisfies (NFLVR).*

In view of this theorem, we will henceforth assume that S is locally bounded and that

Assumption 1 (NFLVR) $\mathcal{M}^e(S) \neq \emptyset$.

The hedging problem can be made specific by introducing the agent's utility $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$. Beginning with initial capital $x \in \mathbb{R}$, the agent then solves the optimal hedging problem

$$\sup_{H \in \mathcal{H}} E\left[U\left(x + (H \cdot S)_T - B\right)\right]. \tag{2}$$

If $B \equiv 0$, the optimal hedging problem reduces to Merton's optimal investment problem

$$\sup_{H \in \mathcal{H}} E\left[U\left(x + (H \cdot S)_T\right)\right]. \tag{3}$$

Assumption 2 *The utility function* $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ *is increasing, differentiable and strictly concave satisfying*

$$\lim_{x \to \infty} U'(x) = 0. \tag{4}$$

Furthermore, we assume that $\lim_{x \to -\infty} U'(x) = \infty$,

$$\limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1 \quad \textit{and} \quad \liminf_{x \to -\infty} \frac{xU'(x)}{U(x)} > 1.$$

Define the conjugate function V as the Legendre transform of the function $-U(-x)$, that is

$$V(y) := \sup_{x \in \mathbb{R}} [U(x) - xy], \qquad y > 0. \tag{5}$$

Proposition 4 *If U satisfies assumption 2, then the conjugate function V is finite valued, differentiable, strictly convex on $(0, \infty)$ and satisfies*

$$\lim_{y \to 0} V(y) = \lim_{x \to \infty} U(x), \qquad \lim_{y \to 0} V'(y) = -\infty. \tag{6}$$

Moreover, the behaviour of V at infinity is

$$\lim_{y \to \infty} V(y) = \infty \qquad \textit{and} \qquad \lim_{y \to \infty} V'(y) = \infty.$$

Define

$$C_U^b(x) = \{g \in L^0(\Omega, \mathcal{F}_T, P) : g \leq x + (H \cdot S)_T \text{ for some } H \in \mathcal{H} \text{ and } U(g) \in L^1(\Omega, \mathcal{F}_T, P)\}. \quad (7)$$

and consider

$$C_U(x) = \Big\{f \in L^0(\Omega, \mathcal{F}_T, P; \mathbb{R} \cup \{\infty\}) : U(f) \text{ is in the } L^1(P)\text{-closure of } \{U(g) : g \in C_U^b(x)\}\Big\}. \quad (8)$$

Theorem 5 (S01) *Suppose that assumptions 1 and 2 are satisfied. Then, for any $x \in \mathbb{R}$ and $y > 0$, the problems*

$$u(x) = \sup_{f \in C_U(x)} E[U(f)], \qquad v(y) = \inf_{Q \in M^a(S)} E\left[V\left(y\frac{dQ}{dP}\right)\right] \quad (9)$$

have unique optimizers $\widehat{f}(x) \in C_U(x)$ and $\widehat{Q}(y) \in \mathcal{M}^a(S)$ satisfying

$$U'(\widehat{f}(x)) = y\frac{d\widehat{Q}(y)}{dP}, \quad (10)$$

where x and y are related by $u'(x) = y$.

Corollary 6 *Let* $U(x) = -\frac{e^{-\gamma x}}{\gamma}$, $\gamma > 0$, *and suppose that assumption 1 holds. If in addition we have that*

$$E\left[\frac{dQ}{dP}\log\left(\frac{dQ}{dP}\right)\right] < \infty, \tag{11}$$

for some $Q \in \mathcal{M}^e(S)$, *then the minimizer* $\widehat{Q}(y)$ *of theorem 5 is the equivalent local martingale measure* $\widehat{Q}$, *independent of* $y > 0$, *which minimizes the relative entropy with respect to P among all absolutely continuous martingale measures. Therefore* $\widehat{f}(x)$ *equals the terminal value* $\widehat{X}_T(x)$ *of a uniformly integrable* $\widehat{Q}$*-martingale of the form*

$$\widehat{X}_t(x) = x + (\widehat{H}(x) \cdot S)_t,$$

for some $\widehat{H}(x) \in L(S)$.

Assumption 3 $B \in L^\infty(\Omega, \mathcal{F}_T, P)$.

Define

$$\mathcal{C}_U^b(x) = \{g \in L^0(\Omega, \mathcal{F}_T, P) : g \le x + (H \cdot S)_T - B \text{ for some } H \in \mathcal{H} \text{ and } U(g) \in L^1(\Omega, \mathcal{F}_T, P)\}. \quad (12)$$

Similarly, we replace the set $C_U(x)$ by

$$\mathcal{C}_U(x) = \Big\{f \in L^0(\Omega, \mathcal{F}_T, P; \mathbb{R} \cup \{\infty\}) : U(f - B) \text{ is in the } L^1(P)\text{-closure of } \{U(g) : g \in C_U^b(x)\}\Big\}. \quad (13)$$

Theorem 7 (0wen02) *Suppose that assumptions 1, 2 and 3 are satisfied. Then, for any $x \in \mathbb{R}$ and $y > 0$, the problems*

$$u(x) = \sup_{f \in \mathcal{C}_U(x)} E[U(f - B)], \quad v(y) = \inf_{Q \in M^a(S)} E\left[V\left(y\frac{dQ}{dP}\right) - y\frac{dQ}{dP}B\right]$$

have unique optimizers $\widehat{f}(x) \in C_U(x)$ and $\widehat{Q}(y) \in \mathcal{M}^a(S)$ satisfying

$$U'(\widehat{f}(x) - B) = y\frac{d\widehat{Q}(y)}{dP}, \tag{14}$$

where x and y are related by $u'(x) = y$.

Corollary 8 *Let* $U(x) = -\frac{e^{-\gamma x}}{\gamma}$, $\gamma > 0$, *and suppose that assumptions 1 and 3 hold. If in addition we have that*

$$E\left[\frac{dQ}{dP}\log\left(\frac{dQ}{dP}\right)\right] < \infty, \tag{15}$$

for some $Q \in \mathcal{M}^e(S)$, *then the minimizer* $\widehat{Q}(y)$ *of theorem 7 is the equivalent local martingale measure* $\widehat{Q}$, *independent of* $y > 0$, *which minimizes the relative entropy with respect to* P_B *among all absolutely continuous martingale measures. Therefore* $\widehat{f}(x)$ *equals the terminal value* $\widehat{X}_T(x)$ *of a uniformly integrable* $\widehat{Q}$*-martingale of the form*

$$\widehat{X}_t(x) = x + (\widehat{H}(x) \cdot S)_t,$$

for some $\widehat{H}(x) \in L(S)$.

2. The dynamics of portfolio selection

For any intermediate time $t \in [0, T]$ and $x \in \mathbb{R}$, we can write

$$u(x) = \sup_{H \in \mathcal{A}_{(0,t]}} E\left[\operatorname*{ess\,sup}_{H \in \mathcal{A}_{(t,T]}} E_t[U(x + (H \cdot S)_0^t + (H \cdot S)_t^T - B)]\right], \tag{16}$$

which leads us to the study of the conditional problem

$$u_t(\mathrm{w}) = \operatorname*{ess\,sup}_{H \in \mathcal{A}_{(t,T]}} E_t[U(\mathrm{w} + (H \cdot S)_t^T - B)], \tag{17}$$

where $\mathrm{w} \in \mathbb{R}$ represents the wealth accumulated up to time t. The dynamic programming principle for this stochastic control problem has the form

$$u_s(\mathrm{w}) = \operatorname*{ess\,sup}_{H \in \mathcal{A}_{(s,t]}} E_s[u_t(\mathrm{w} + (H \cdot S)_s^t)], \tag{18}$$

for $0 \leq s \leq t \leq T$.

The certainty equivalent value and the indifference price

Define

$$B_t(\mathsf{w}) = \mathsf{w} - U^{-1}(u_t(\mathsf{w})), \tag{19}$$

which can be called the certainty equivalent value of the claim B at time t, since

$$U(\mathsf{w} - B_t(\mathsf{w})) = E_t[U(\mathsf{w} + (\widehat{H}^{(\mathsf{w},t)} \cdot S)_t^T - B)].$$

Consider an investor who, holding wealth w at time t, faces the two scenarios:

- sell a claim B for the price π, hedge optimally and achieve

$$E_t[U(\mathrm{w} + \pi + (\widehat{H}^{(\mathrm{w}+\pi,t)} \cdot S)_t^T - B)] = U(\mathrm{w} + \pi - B_t(\mathrm{w} + \pi))$$

- don't sell the claim, invest optimally and achieve

$$E_t[U(\mathrm{w} + (\widehat{H}^{(\mathrm{w},t)}(0) \cdot S)_t^T)] = U(\mathrm{w} - B_t^0(\mathrm{w})).$$

The indifference price of the claim B at time t for wealth w is the value for $\pi = \pi_t^B(\mathrm{w})$ which makes these equal, that is, it is the solution of

$$\pi_t^B(\mathrm{w}) = B_t(\mathrm{w} + \pi_t^B(\mathrm{w})) - B_t^0(\mathrm{w}). \tag{20}$$

The Davis price

If, for each $\varepsilon \geq 0$, we let $B_t^\varepsilon(\mathsf{w})$ denote the certainty equivalent value of εB, then the Davis price of B is defined to be

$$\pi_t^{Davis}(\mathsf{w}) = \left.\frac{dB_t^\varepsilon(\mathsf{w})}{d\varepsilon}\right|_{\varepsilon=0}. \tag{21}$$

By differentiating the identity

$$U(\mathsf{w} - B_t^\varepsilon(\mathsf{w})) = E_t[U(\mathsf{w} + (\widehat{H}^{(\varepsilon,\mathsf{w},t)} \cdot S)_t^T - \varepsilon B)]$$

at $\varepsilon = 0$, it can be shown that

$$\pi_t^{Davis}(\mathsf{w}) = E_{t,\widehat{Q}_t(y)}[B]. \tag{22}$$

Exponential utility

An important simplification occurs if we specialize to the exponential utility $U(x) = -\frac{e^{-\gamma x}}{\gamma}$, $\gamma > 0$, since $u_t(\mathsf{w})$ factorizes as

$$u_t(\mathsf{w}) = -\frac{e^{-\gamma \mathsf{w}}}{\gamma} \operatorname*{ess\,inf}_{H \in \mathcal{A}_{(t,T]}} E_t\left[e^{-\gamma (H \cdot S)_t^T + \gamma B}\right] =: -\frac{e^{-\gamma \mathsf{w}}}{\gamma} v_t. \qquad (23)$$

Here we see that v_t is a time dependent but wealth independent $\mathcal{F}_t$–measurable random variable. We also see that the certainty equivalent value

$$B_t = -\frac{1}{\gamma} \log v_t, \qquad (24)$$

the optimal portfolio $\widehat{H}^{(t)}$ and the indifference price π_t are all wealth independent processes.

3. Discrete time hedging

We now restrict to discrete time hedging, where the portfolio processes have the form

$$H_t = \sum_{k=1}^{K} \; H_k \; \mathbf{1}_{(t_{k-1},t_k]}(t) \tag{25}$$

where each H_k is an $\mathbb{R}^d$–valued $\mathcal{F}_{k-1}$ random variable, with a uniform partition of the interval $[0,T]$ into K subintervals. Now the dynamic programming problem (18) falls into K subproblems

$$u_{k-1}(x) = \operatorname*{ess\,sup}_{H \in \mathcal{A}_{(t_{k-1},t_k]}} \; E_{k-1}[u_k(x + H_k \Delta S_k)], \tag{26}$$

subject to the terminal condition $u_K(x) = U(x)$.

Assumption 4 *The market is Markovian and its state variables* $Z = (S^1, \ldots, S^d, Y^1, \ldots, Y^{n-d})$ *lie in a finite dimensional state space* $\mathcal{S} \in \mathbb{R}^n$*.*

Assumption 5 *The contingent claim is taken to be of the form* $B_T = \Phi(Z_T)$ *for a bounded Borel function* $\Phi : \mathcal{S} \to \mathbb{R}$*.*

In the Markovian setting and for exponential utility, the solution of (26) and the optimal allocation have the wealth independent form

$$
\begin{aligned}
u_k(x) = U(x)v_k &= U(x)g_k(Z_k) && (27)\\
\widehat{H}_{k+1} &= h_{k+1}(Z_k) && (28)\\
B_k &= b_k(Z_k) && (29)
\end{aligned}
$$

for deterministic functions g_k, h_{k+1}, and b_k on the state space $\mathcal{S}$. The iteration equation is simply

$$g_k(Z) = \inf_{h \in \mathbb{R}^d} E_k[\exp(-\gamma h \cdot \Delta S_{k+1}) g_{k+1}(Z_{k+1}) | Z_k = Z] \qquad (30)$$

and the optimal h defines the function $h_{k+1}(Z)$.

4. The exponential utility allocation algorithm

1. Step $k = K$**:** The final optimal allocation $\widehat{H}_K$ is defined to be the $\mathbb{R}^d$–valued $\mathcal{F}_{K-1}$ random variable which solves

$$\min_{H \in \mathcal{A}_{(K-1,K]}} E[\exp(-H \cdot \Delta S_K + B)] \tag{31}$$

Since the solution is of the form $\widehat{H}_K = h_K(Z_{K-1})$, we write this as

$$\min_{h \in \mathcal{B}(\mathcal{S})} \Psi_K(h) \tag{32}$$

where $\Psi_K(h) := E[\exp(-h(Z_{K-1}) \cdot \Delta S_K + B)]$.

Now pick an R–dimensional subspace $\mathcal{R}(\mathcal{S}) \subset \mathcal{B}(\mathcal{S})$ of functions on $\mathcal{S}$ and attempt to "learn" a suboptimal solution

$$h_K^{\mathcal{R}} = \arg\min_{h \in \mathcal{R}(\mathcal{S})} \Psi_K(h)$$

Approximated the expectation $\Psi_K(h)$ by the finite sample estimate

$$\widetilde{\Psi}_K(h) = \frac{1}{N} \sum_{i=1}^{N} \exp\left(-h(Z_{K-1}^i) \cdot \Delta S_K^i + \Phi(Z_K^i)\right) \tag{33}$$

This leads to the estimator $\tilde{h}_K^{\mathcal{R}}$ based on $\{Z_k^i\}$ and the choice of subspace $\mathcal{R}$ defined by

$$\tilde{h}_K^{\mathcal{R}} = \arg\min_{h \in \mathcal{R}(\mathcal{S})} \widetilde{\Psi}_K(h) \tag{34}$$

2. Inductive step for $k = K-1, \ldots, 2$: The estimate $\widetilde{h}_k^{\mathcal{R}}$ of the optimal rule $\widehat{h}_k$, for $2 \leq k < K-1$ is determined inductively given the estimates $\widetilde{h}_{k+1}^{\mathcal{R}}, \ldots, \widetilde{h}_K^{\mathcal{R}}$. It is defined to be

$$\widetilde{h}_k^{\mathcal{R}} = \operatorname*{arg\,min}_{h \in \mathcal{R}(\mathcal{S})} \widetilde{\Psi}_k(h; \widetilde{h}_{k+1}^{\mathcal{R}}, \ldots, \widetilde{h}_K^{\mathcal{R}}) \tag{35}$$

where

$$\widetilde{\Psi}_k(h; \widetilde{h}_{k+1}^{\mathcal{R}}, \ldots, \widetilde{h}_K^{\mathcal{R}}) = \frac{1}{N} \sum_{i=1}^{N} \exp\left(-h(Z_{k-1}^i) \cdot \Delta S_k^i - \sum_{j=k+1}^{K} \widetilde{h}_j^{\mathcal{R}}(Z_{j-1}^i) \cdot \Delta S_j^i + \Phi(Z_K^i) \right)$$

3. Final step $k = 1$: This step is degenerate since the initial values Z_0 are constant over the sample. Therefore we determine the optimal constant vector $\tilde{h}_1 \in \mathbb{R}^d$ by solving

$$\tilde{h}_1 = \arg\min_{h \in \mathbb{R}^d} \widetilde{\Psi}_1(h; \tilde{h}_2^{\mathcal{R}}, \dots, \tilde{h}_K^{\mathcal{R}}) \tag{36}$$

To summarize, the algorithm above learns a collection of functions of the form $(\tilde{h}_1, \tilde{h}_2^{\mathcal{R}}, \dots, \tilde{h}_K^{\mathcal{R}}) \in \mathbb{R}^d \times \mathcal{R}(\mathcal{S})^{K-1}$ from the Monte Carlo simulation. This collection defines a suboptimal allocation strategy for the exponential hedging problem. Finally, the optimal value $\widetilde{\Psi}_1(\tilde{h}_1; \tilde{h}_2^{\mathcal{R}}, \dots, \tilde{h}_K^{\mathcal{R}})$ is an estimate of the quantity e^{B_0}, where B_0 is the certainty equivalent value of the claim B at time $t = 0$.

5. Numerical implementation

Consider a one-dimensional GBM with parameters $S_0 = 1$, $\mu = 0.1$, $\sigma = 0.2$ and $r = 0.0$ over the period of one year $T = 1$.

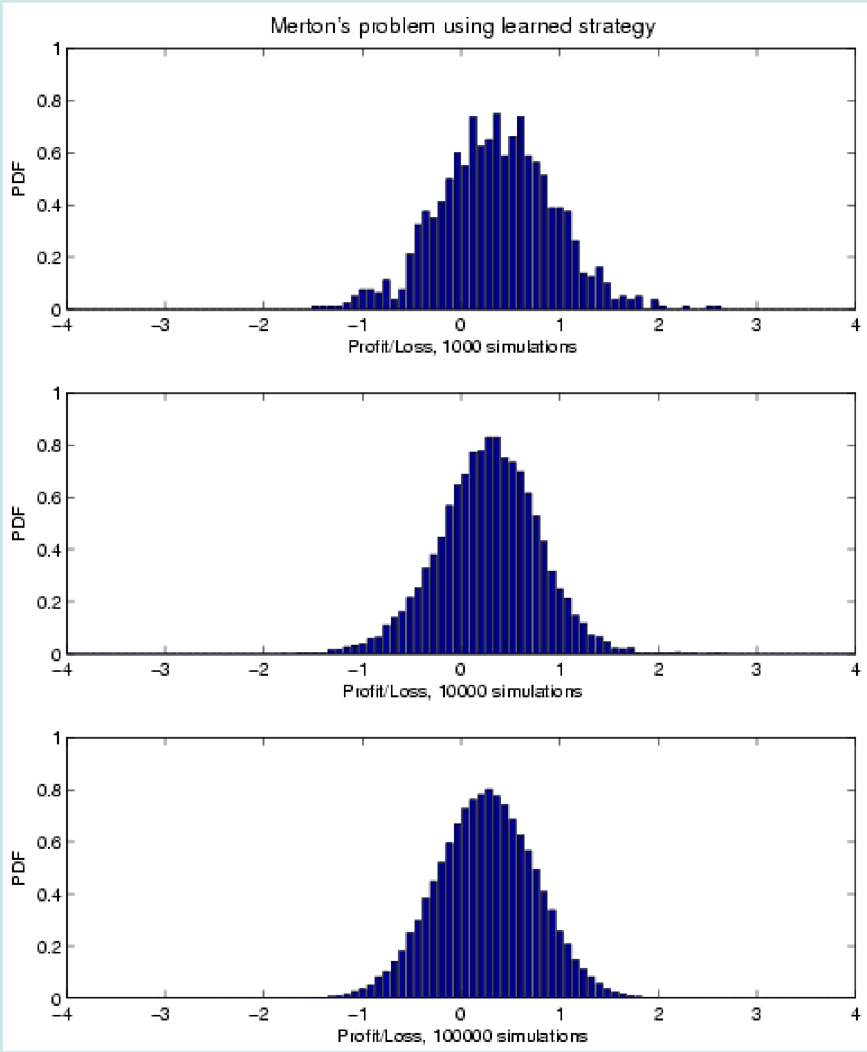
Apply the allocation algorithm with $K = 50$ (i.e. weekly) to

1. the Merton investment problem

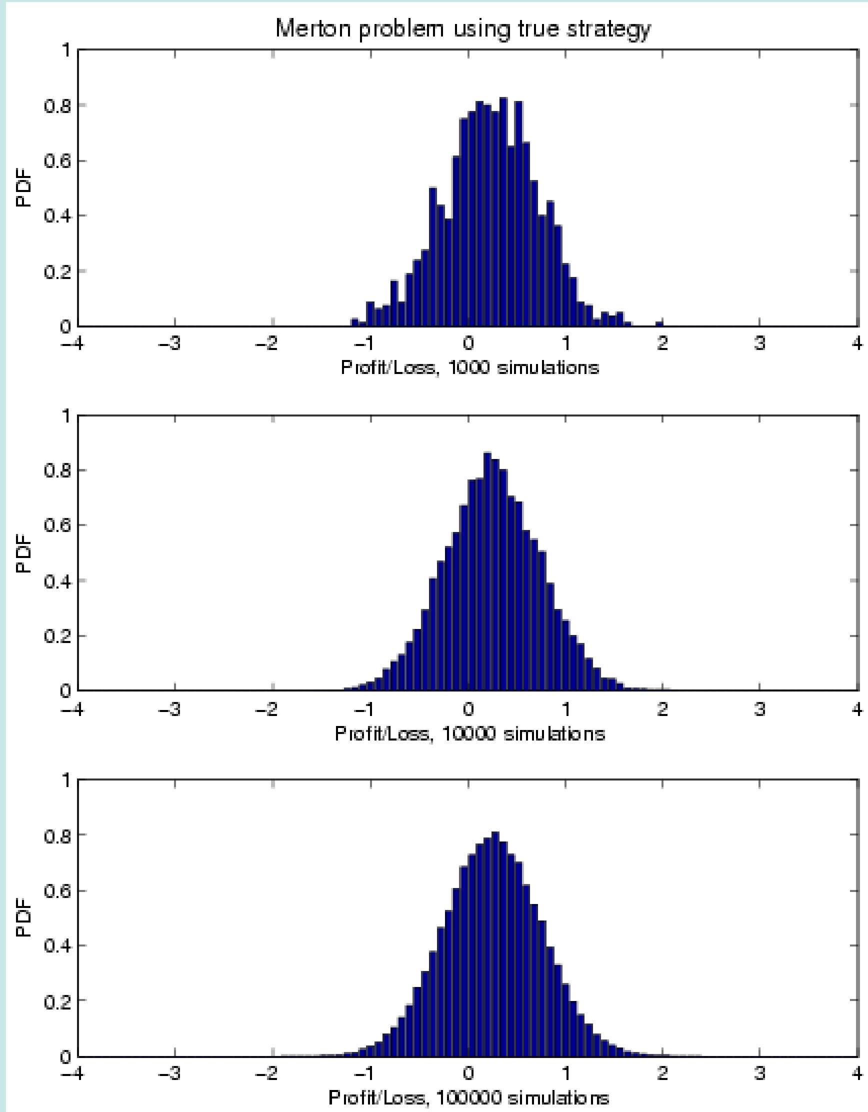
2. the hedging problem for the buyer of an European put

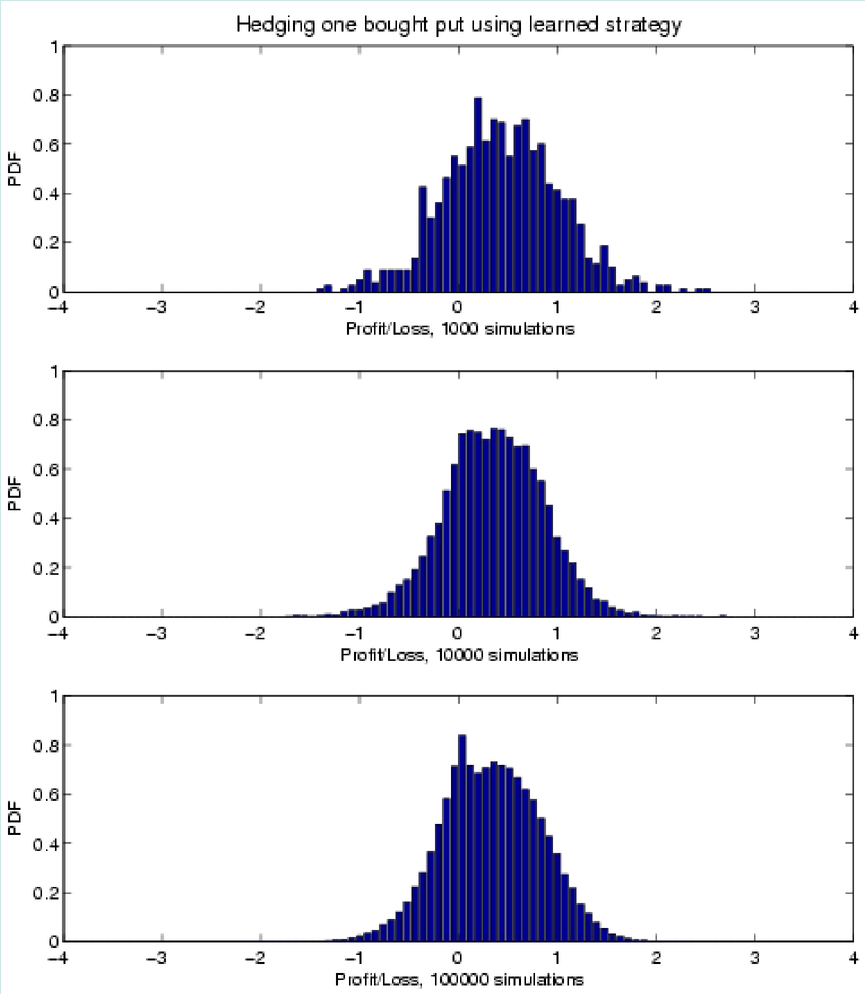
In each case, set $N = 1000, 10000$ and 100000.

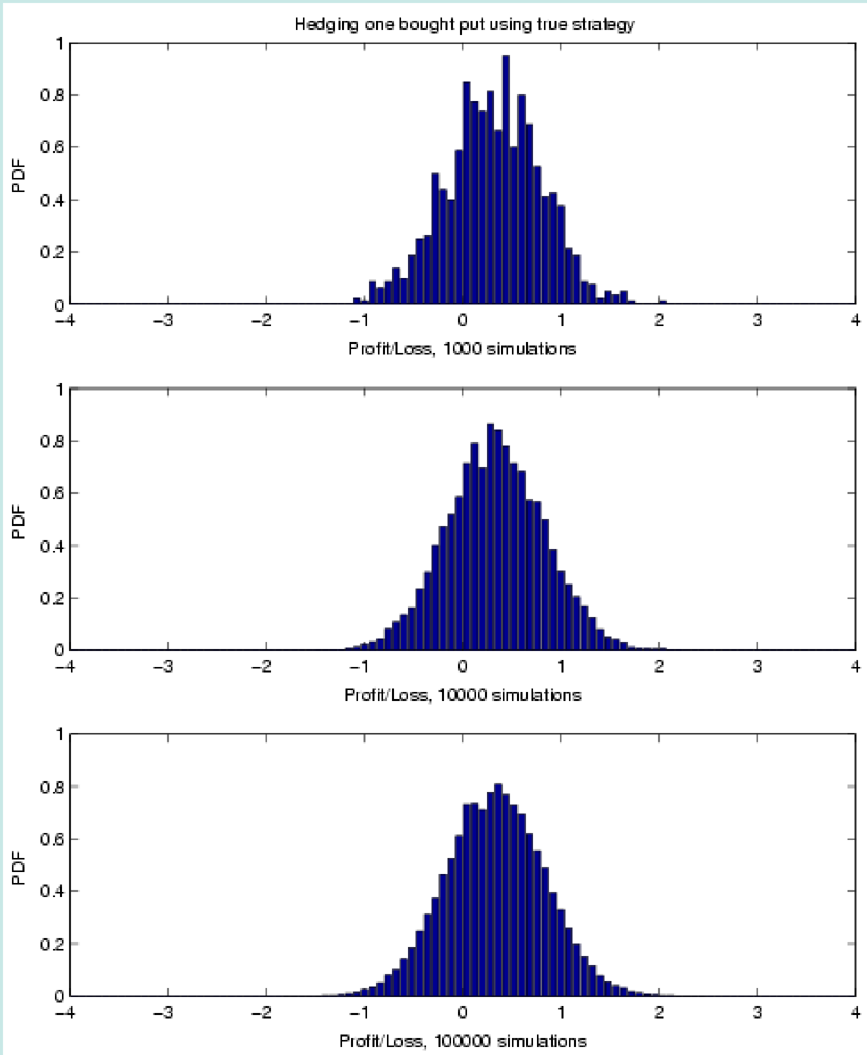
For comparison to theory, use the same Monte Carlo simulations, but rehedged weekly according to the theoretical Black-Scholes delta of the option.

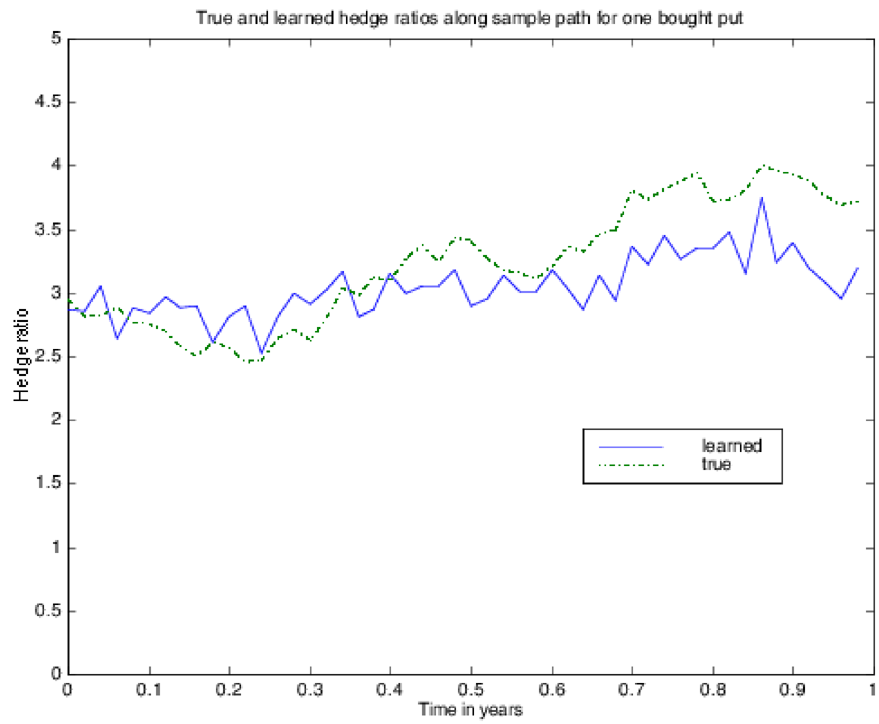


Merton's problem using learned strategy
PDF
Profit/Loss, 1000 simulations
PDF
Profit/Loss, 10000 simulations
PDF
Profit/Loss, 100000 simulations


Merton problem using true strategy
PDF
Profit/Loss, 1000 simulations
PDF
Profit/Loss, 10000 simulations
PDF
Profit/Loss, 100000 simulations


Hedging one bought put using learned strategy
PDF
Profit/Loss, 1000 simulations
PDF
Profit/Loss, 10000 simulations
PDF
Profit/Loss, 100000 simulations


Hedging one bought put using true strategy
PDF
Profit/Loss, 1000 simulations
PDF
Profit/Loss, 10000 simulations
PDF
Profit/Loss, 100000 simulations


True and learned hedge ratios along sample path for one bought put
Hedge ratio
Time in years
learned
true
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

The learned estimates of the indifference price are $0.0767, 0.0790$ and 0.0792 for the cases $N = 1000, 10000$ and 100000.

Using the true strategy leads to the values $0.0798, 0.0796$ and 0.0795, respectively.

The theoretical Black-Scholes price is 0.0797.