Nonlinearity, correlation and the valuation of employee options

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We need a little controversy...

- In 1999, Warren Buffet famously asked:

  If options aren't a form of compensation, what are they? If compensation isn't an expense, what is it? And if expenses shouldn't go into the calculation of earnings, where in the world should they go?

- The categorical answer came from Z.Bodie and R.Merton in March 2003:

  For the Last Time: Stock Options Are an Expense

- Not for Craig Barret, however (Congress hearing, June 2003):

  With all due respect to those who would support option expensing, I suggest they focus their efforts on fixing the current shortcomings of our accounting principles before they move to take away something that underpins our economic competitiveness.

Accounting reccomendations

- The *Financial Accounting Standard Board* instructed in 1972 (Opinion 25) that stock options should be accounted according to their intrinsic value, that is $(Y_t - K)^+$ on the date their are granted.
- In 1995, the FASB 123 recommended using a fair value approach instead: estimate the expected life of the option and insert this into either Black–Scholes or a Cox–Rubenstein-Ross tree. It still accepted Opinion 25 as a valid method.
- In 2004, it revised FASB 123, eliminating the possibility of using intrinsic value methods.

Previous literature

- Detemple and Sudaresan (1999) and Hall and Murphy (2002) propose to use utility methods to deal with the market incompleteness created by trading and hedging restrictions, but without using a correlated asset.
- Rogers and Scheinkman (2003) and Jain and Subramanian (2004) investigate the effect of partial exercise, but with no correlated asset.
- Hull and White (2004) use a binomial model with no correlated asset, no partial exercise and no risk preferences. The incompleteness is accounted for by a parameter M - the effective stock-to-strike exercise threshold.
- Cvitanic et al (2004) and Sircar and Wei (2005) develope continuous–time versions for similar models.
- Henderson (2005) applied indifference pricing to value a single American call options on a non-traded asset, with infinite time horizon.

Contributions of this paper

We propose a valuation procedure that:

- is FASB complaisant;
- is implemented in discrete–time within a finite time horizon;
- allows (but does not require) trade in a correlated asset;
- takes into account the presence of multiple claims;
- resolves market incompleteness by consistently incorporating risk preferences.

The Problem

We consider an employee who has been awarded a compensation package consisting of A identical call options on the company's stock with the following features:

- strike price K, maturity date T;
- options are non-transferible;
- hedge using the underlying stock Y_t is not allowed;
- hedge using a correlated asset S_t is allowed.

The one-period model

Consider a one-period market model where discounted prices are given by

$$(S_T, Y_T) = \begin{cases} (uS_0, hY_0) & \text{with probability } p_1, \\ (uS_0, \ell Y_0) & \text{with probability } p_2, \\ (dS_0, hY_0) & \text{with probability } p_3, \\ (dS_0, \ell Y_0) & \text{with probability } p_4, \end{cases} \tag{1}$$

where $0 < d < 1 < u$ and $0 < \ell < 1 < h$, for positive initial values S_0, Y_0 and historical probabilities p_1, p_2, p_3, p_4
We assume that risk preferences are given by an exponential utility function $U(x) = -e^{-\gamma x}$.

Optimal hedge and the indifference price

Let $C_T = C(Y_T)$ be a the discounted payoff at time T. An investor who buys this claim for a price π will then try to solve the optimal portfolio problem

$$u^C(x-\pi) = \sup_H E[U(X_T + C_T)], \tag{2}$$

where $X_T = x + H(S_T - S_0)$ is the discounted terminal wealth. The indifference price for this claim is defined to be a solution to the equation

$$u^0(x) = u^C(x-\pi),$$

where u^0 is defined by (2) for the degenerate case $C \equiv 0$.

An expression for the Indifference Price

Explicit calculations then lead to

$$\pi = g(C_h, C_\ell) \tag{3}$$

where, for fixed parameters $(u, d, p_1, p_2, p_3, p_4)$ the function $g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$g(x_1, x_2) = \frac{q}{\gamma} \log\left(\frac{p_1 + p_2}{p_1 e^{-\gamma x_1} + p_2 e^{-\gamma x_2}}\right) + \frac{1-q}{\gamma} \log\left(\frac{p_3 + p_4}{p_3 e^{-\gamma x_1} + p_4 e^{-\gamma x_2}}\right)$$

with

$$q = \frac{1-d}{u-d}.$$

Early exercise

Now suppose C is an American claim. It is clear that early exercise will occur whenever

$$C(Y_0) \geq \pi,$$

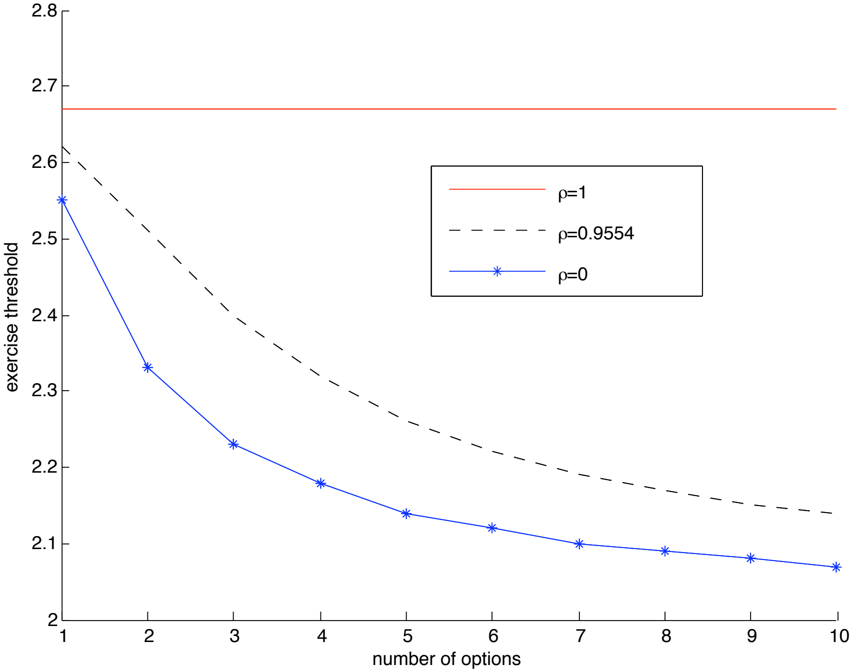
where π^B is the (European) indifference price. For example, an American call option with strike price K will be exercised if Y_0 exceeds the solution to

$$(Y - K)^+ = g((hY - e^{-rT}K)^+, (\ell Y - e^{-rT}K)^+)$$

Multiple claims

As a result of risk aversion, the early exercise threshold for one American call option obtained above is different (and higher) than the exercise threshold for a contract consisting of A units of identical Americal calls. Explicitly, it is the solution to

$$A(Y-K)^{+}=g(A(hY-e^{-rT}K)^{+},A(\ell Y-e^{-rT}K)^{+}) \qquad (4)$$



2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2
1
2
3
4
5
6
7
8
9
10
exercise threshold
number of options
ρ=1
ρ=0.9554
ρ=0

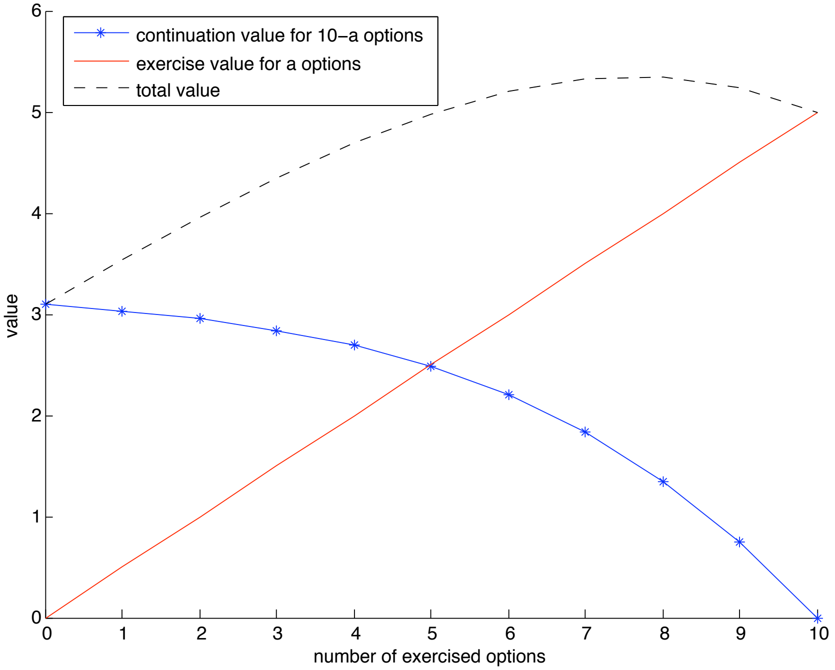
Partial Exercise

- If partial exercise is allowed, then the optimal number of options to be exercised is the solution a^* to

$$\max_a \left[a(Y_0 - K)^+ + \pi^{(A-a)} \right]. \tag{5}$$

- The value of A units of the option is therefore

$$C_0^{(A)} = a_0(Y_0 - K)^+ + \pi^{(A-a_0)}$$


continuation value for 10−a options
exercise value for a options
total value
value
number of exercised options
0
1
2
3
4
5
6
7
8
9
10
0
1
2
3
4
5
6

Two-period model: inter-temporal exercise

- Let us label the nodes in of a two-period binomial tree by 0 at time $t_0 = 0$, (h, ℓ) at time t_1 and $(hh, h\ell, \ell\ell)$ at time $t_2 = T$.
- The number of option that the holder of A calls at the node h should immediately exercise is given by

$$a_h = \arg\max_{0 \leq a \leq A} \left[a(hY_0 - e^{-rT/2}K)^+ + \pi_h^{(A-a)} \right], \qquad (6)$$

where $\pi_h^{(A-a)}$ denotes the indifference of an European claim to starting at the node h and maturing at time T.

- The possible pay-offs for such claim are

$$C_{hh}^{(A-a)} = (A-a)(hhY_0 - K)^+$$

with probability $(p_1 + p_3)$ and

$$C_{h\ell}^{(A-a)} = (A-a)(h\ell Y_0 - K)^+$$

with probability $(p_2 + p_4)$, where we have used hh and $h\ell$ to denoted, respectively, the nodes where the non-traded asset has values hhY_0 and $h\ell Y_0$.
- Its indifference price is explicitly given by

$$\pi_h^{(A-a)} = g(C_{hh}^{(A-a)}, C_{h\ell}^{(A-a)}) \tag{7}$$

- Similarly, the optimal number of options to be exercised at the node ℓ, where the non-trade asset has value ℓY_0, is

$$a_\ell = \arg\max_{0 \le a \le A} \left[a(\ell Y_0 - K)^+ + \pi_\ell^{(A-a)} \right], \tag{8}$$

where

$$\pi_\ell^{(A-a)} = g(C_{h\ell}^{(A-a)}, C_{\ell\ell}^{(A-a)}) \tag{9}$$

- Therefore, at the intermediate time t_1, the total value of A options at the node h is

$$C_h^{(A)} := \left[a_h(hY_0 - K)^+ + \pi_h^{(A-a_h)} \right], \tag{10}$$

while the total value of A options at the node ℓ is

$$C_\ell^{(A)} := \frac{1}{A} \left[a_\ell(\ell Y_0 - K)^+ + \pi_\ell^{(A-a_\ell)} \right]. \tag{11}$$

- Finally, the number of options that should be exercised at the initial time t_0 is

$$a_0 = \arg \max_{0 \leq a \leq A} \left[a(Y_0 - K)^+ + \pi_0^{(A-a)} \right], \tag{12}$$

where

$$\pi_h^{(A-a)} = g(C_h^{(A-a)}, C_\ell^{(A-a)}) \tag{13}$$

- Therefore the value at time zero of A units of an American call option on the non-traded asset is

$$C_0^{(A)} := \left[a_0(Y_0 - K)^+ + \pi_0^{(A-a_0)} \right]. \tag{14}$$

The multi-period model

- We first have to choose discrete time parameters $(u, d, h, \ell, p_1, p_2, p_3, p_4)$ that match the distributional properties of the continuos time diffusion

$$
\begin{aligned}
dS &= (\mu - r)Sdt + \sigma SdW & (15)\\
dY &= (a - r - \delta)Ydt + bY(\rho dW + \sqrt{1-\rho^2})dZ, & (16)
\end{aligned}
$$

- These are given by the system

$$
\begin{aligned}
u &= e^{\sigma\sqrt{\Delta t}}, \qquad h = e^{b\sqrt{\Delta t}}\\
d &= e^{-\sigma\sqrt{\Delta t}}, \qquad \ell = e^{-b\sqrt{\Delta t}}\\
p_1 + p_2 &= \frac{e^{(\mu-r)\Delta t} - d}{u - d}\\
p_1 + p_3 &= \frac{e^{(a-r-\delta)\Delta t} - \ell}{h - \ell}\\
\rho b \sigma \Delta t &= (u-d)(h-\ell)[p_1 p_4 - p_2 p_3]\\
1 &= p_1 + p_2 + p_3 + p_4
\end{aligned}
$$

The valuation algorithm

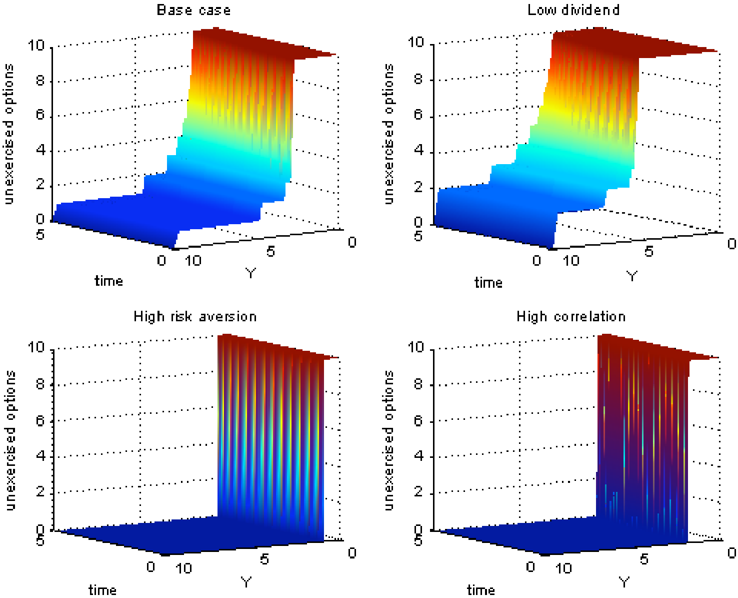
- Begin at the final period.
- At each node of the tree, compute the (European) indifference prices for different values of $(A - a)$.
- Determining the maximum of (5).
- Use this as the value for the entire position at that node.
- Iterate backwards.

Exercise Surface

- We first determine the optimal exercise surface for the holder of $A = 10$ options with strike price $K = 1$ and

$$\begin{aligned} \mu &= 0.12, \quad \sigma = 0.2, \quad S_0 = 1 & (17) \\ a &= 0.15 \quad b = 0.3, \quad Y_0 = 1 & (18) \\ r &= 0.06 \quad T = 5, \quad N = 500 & (19) \end{aligned}$$

- For our base case, $\delta = 0.075$, $\gamma = 0.125$ and $\rho = -0.5$. We then modify it by having $\delta = 0$, $\gamma = 10$ and $\rho = 0.95$.



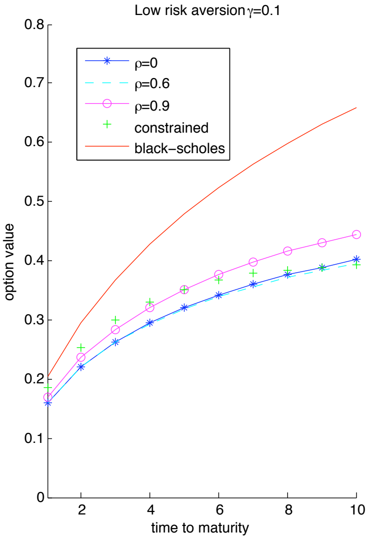
Base case
Low dividend
High risk aversion
High correlation
unexercised options
time
Y
0
2
4
6
8
10
5

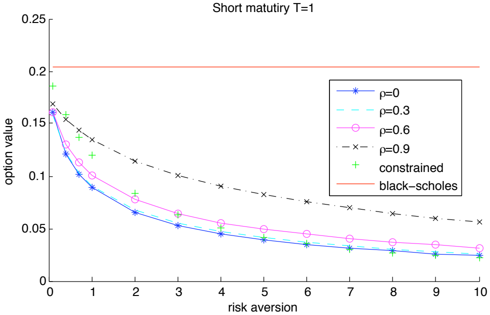
Option value

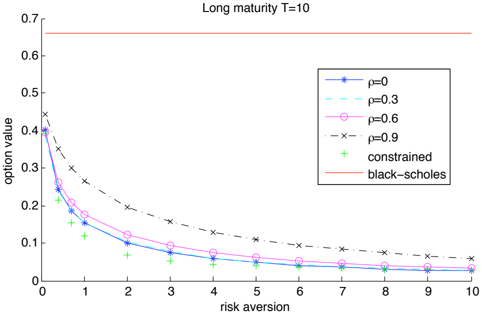
- Next we consider the impact that time-to-maturity, risk aversion, correlation and volatility have on the option price, using the parameters

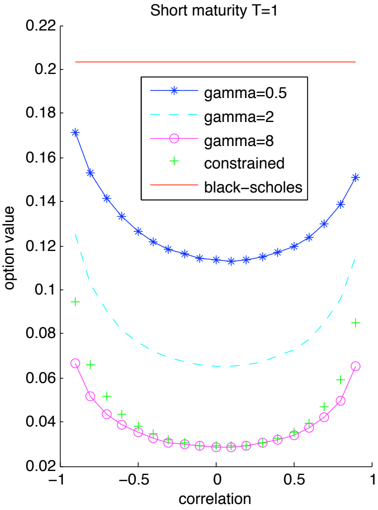
$$\begin{aligned} \mu &= 0.09, \quad \sigma = 0.4, \quad S_0 = 1 && (20) \\ a &= 0.08 \quad b = 0.45, \quad Y_0 = 1 && (21) \\ r &= 0.06 \quad \delta = 0, \quad N = 100 && (22) \end{aligned}$$

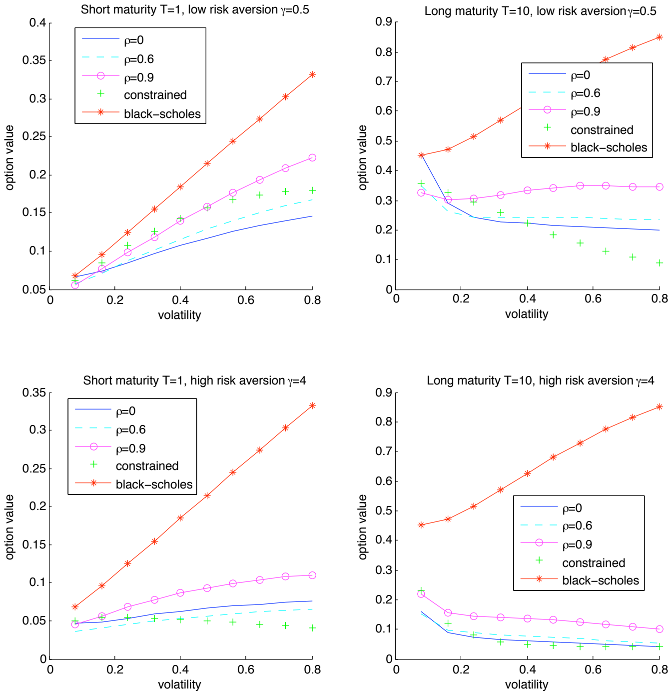
- For comparison, we also plot the corresponding Black–Scholes price (complete market), as well as the value obtained if all options are exercised at once (constrained model).
- When not indicated, the constrained model uses $\rho = 0.9$ and $\gamma = 2$.


Low risk aversion γ=0.1
Higher risk aversion γ=2
option value
time to maturity
ρ=0
ρ=0.6
ρ=0.9
constrained
black−scholes
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
2
4
6
8
10


Short matutiry T=1
option value
risk aversion
0.25
0.2
0.15
0.1
0.05
0
0 1 2 3 4 5 6 7 8 9 10
ρ=0
ρ=0.3
ρ=0.6
ρ=0.9
constrained
black−scholes

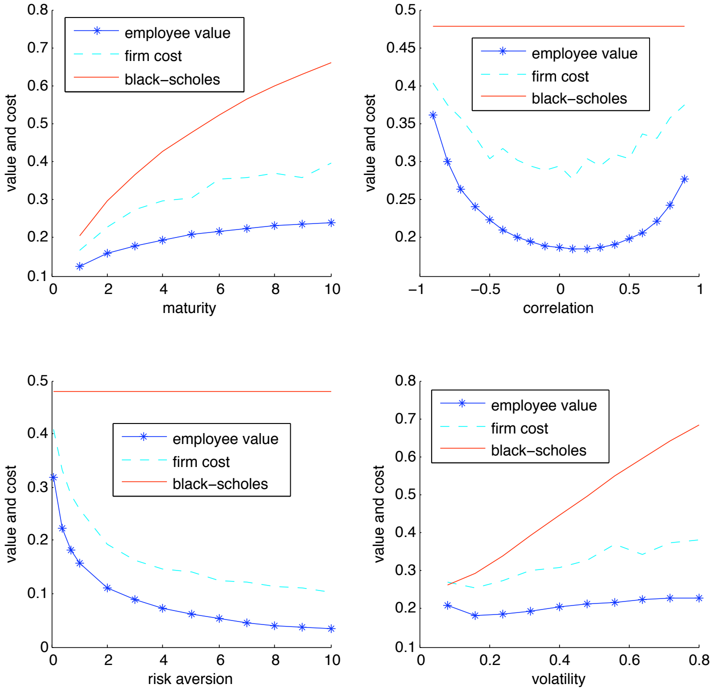

Long maturity T=10
option value
risk aversion
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8 9 10
ρ=0
ρ=0.3
ρ=0.6
ρ=0.9
constrained
black−scholes


Short maturity T=1
Long maturity T=10
option value
correlation
gamma=0.5
gamma=2
gamma=8
constrained
black–scholes
0.22
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−1
−0.5
0
0.5
1


Short maturity T=1, low risk aversion γ=0.5
option value
volatility
ρ=0
ρ=0.6
ρ=0.9
constrained
black−scholes
Long maturity T=10, low risk aversion γ=0.5
option value
volatility
ρ=0
ρ=0.6
ρ=0.9
constrained
black−scholes
Short maturity T=1, high risk aversion γ=4
option value
volatility
ρ=0
ρ=0.6
ρ=0.9
constrained
black−scholes
Long maturity T=10, high risk aversion γ=4
option value
volatility
ρ=0
ρ=0.6
ρ=0.9
constrained
black−scholes

Cost for the firm

- We assume that the firm is well–diversified and faces no trade restrictions.
- Therefore, the cost of issuing an employee option is obtained as the discounted risk–neutral expected payoff for the option at the exercise dates.
- We obtain this by simulating the risk–neutral dynamics for the stock Y_t, then calculating the optimal exercise policy for the employee along each path (based on a discrete grid), followed by a Monte Carlo average over all paths.



employee value
firm cost
black–scholes
value and cost
maturity
correlation
risk aversion
volatility

Conclusions

- Option values are much lower than the Black–Scholes price.
- Allowing for trade in a correlated asset significantly increases the value for the employee and the cost for the firm.
- Ignoring partial exercise is highly non-optimal.
- Method can be easily extended to incorporate a vesting period and exit rates for employees.