Macroeconomic modelling with heterogeneous agents: the master equation approach

M. R. Grasselli

Mathematics and Statistics - McMaster University

Econophysics Colloquium, ICTP-SAIFR, July 27, 2016

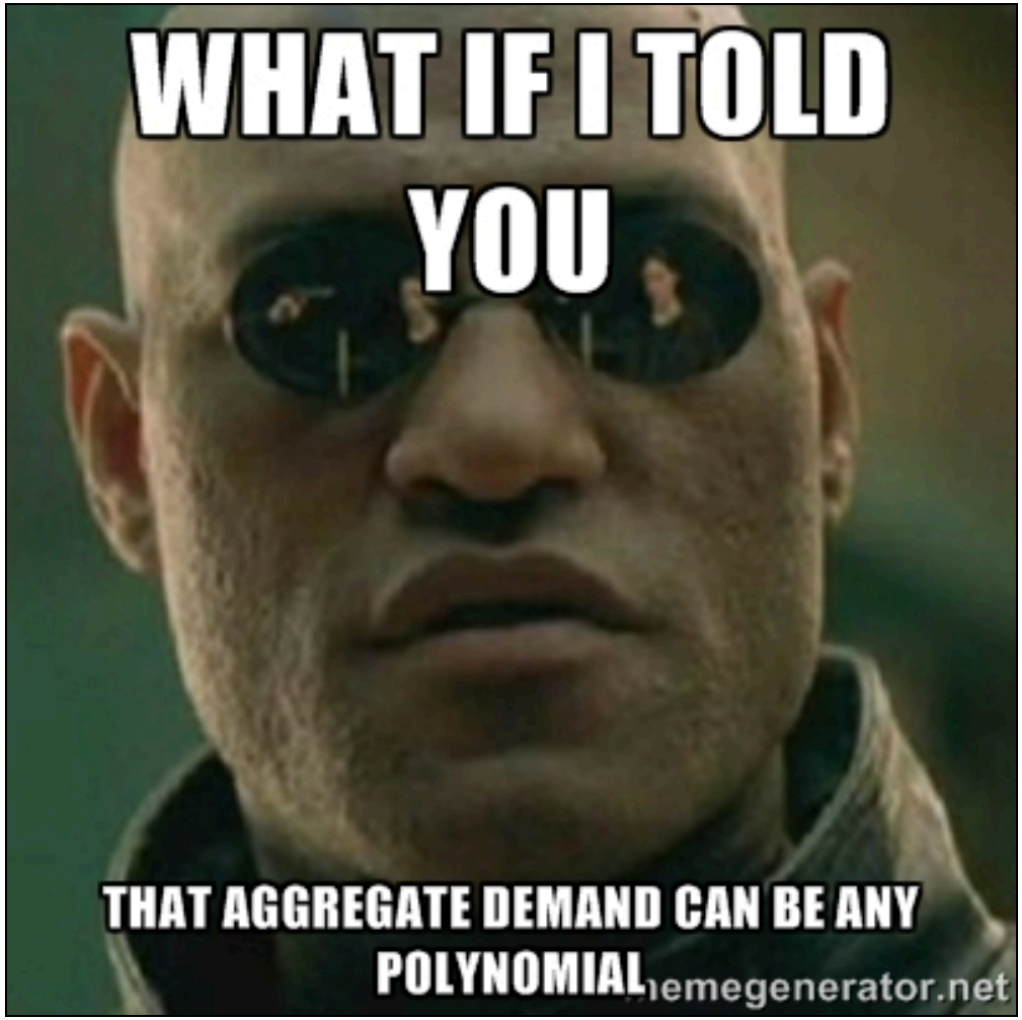
A brief history of Macroeconomics

- Classics (Smith, Ricardo, Marx): no distinction between micro and macro, Say's law, emphasis on long run.
- Beginning of the 20th century (Wicksell, Fisher): natural rate of interest, quantity theory of money.
- Keynesian revolution (1936): shift to demand, fallacies of composition, role of expectations, and much more!
- Neoclassical synthesis - 1945 to 1970 (Hicks, Samuelson, Solow): Keynesian consensus.
- Rational Expectations Revolution - 1972 (Lucas, Prescott, Sargent): internal consistency, microfoundations.
- Start of Macro Wars: Real Business Cycles versus New Keynesian.
- 1990's: impression of consensus around DSGE models, but with different flavours.

Dynamic Stochastic General Equilibrium

- Seeks to explain the aggregate economy using theories based on strong microeconomic foundations.
- Collective decisions of rational individuals over a range of variables for both present and future.
- All variables are assumed to be simultaneously in equilibrium.
- The only way the economy can be in disequilibrium at any point in time is through decisions based on wrong information.
- Money is neutral in its effect on real variables.

SMD theorem: something is rotten in GE land



The SMD theorems

- The following statements are taking from the Shafer and Sonnenschein survey article in *Handbook of Mathematical Economics* (Arrow et al, 1993) summarizing the Sonnenshein, Mantel, Debreu results.
- When preferences are homothetic and the distribution of income is independent of prices, then the market demand function has all the properties of a consumer demand function.
- With general preferences, even if the distribution of income is fixed, market demand functions need not satisfy in any way the classical restrictions which characterize consumer demand functions.

Voices of discontent

- M. Morishima (1984): "If economists successfully devise a correct general equilibrium model (...) should it lack the institutional backing to realize an equilibrium solution, then [it] will amount to no more than a utopian state of affairs which bears no relation whatsoever to the real economy."
- A. Kirman (1989): "[DSGE is] empty in the sense that one cannot expect it to house the elements of a scientific theory, one capable of producing empirically falsifiable propositions".
- K. Arrow (1986): "In the aggregate, the hypothesis of rational behavior has in general no implications."
- R. Solow (2006): "Maybe there is in human nature a deep-seated perverse pleasure in adopting and defending a wholly counterintuitive doctrine that leaves the uninitiated peasant wondering what planet he or she is on."

Really bad economics: hardcore (freshwater) DSGE

- The strand of DSGE economists affiliated with RBC theory made the following predictions after 2008:
  1. Increases government borrowing would lead to higher interest rates on government debt because of "crowding out".
  2. Increases in the money supply would lead to inflation.
  3. Fiscal stimulus has zero effect in an ideal world and negative effect in practice (because of decreased confidence).

Wrong prediction number 1

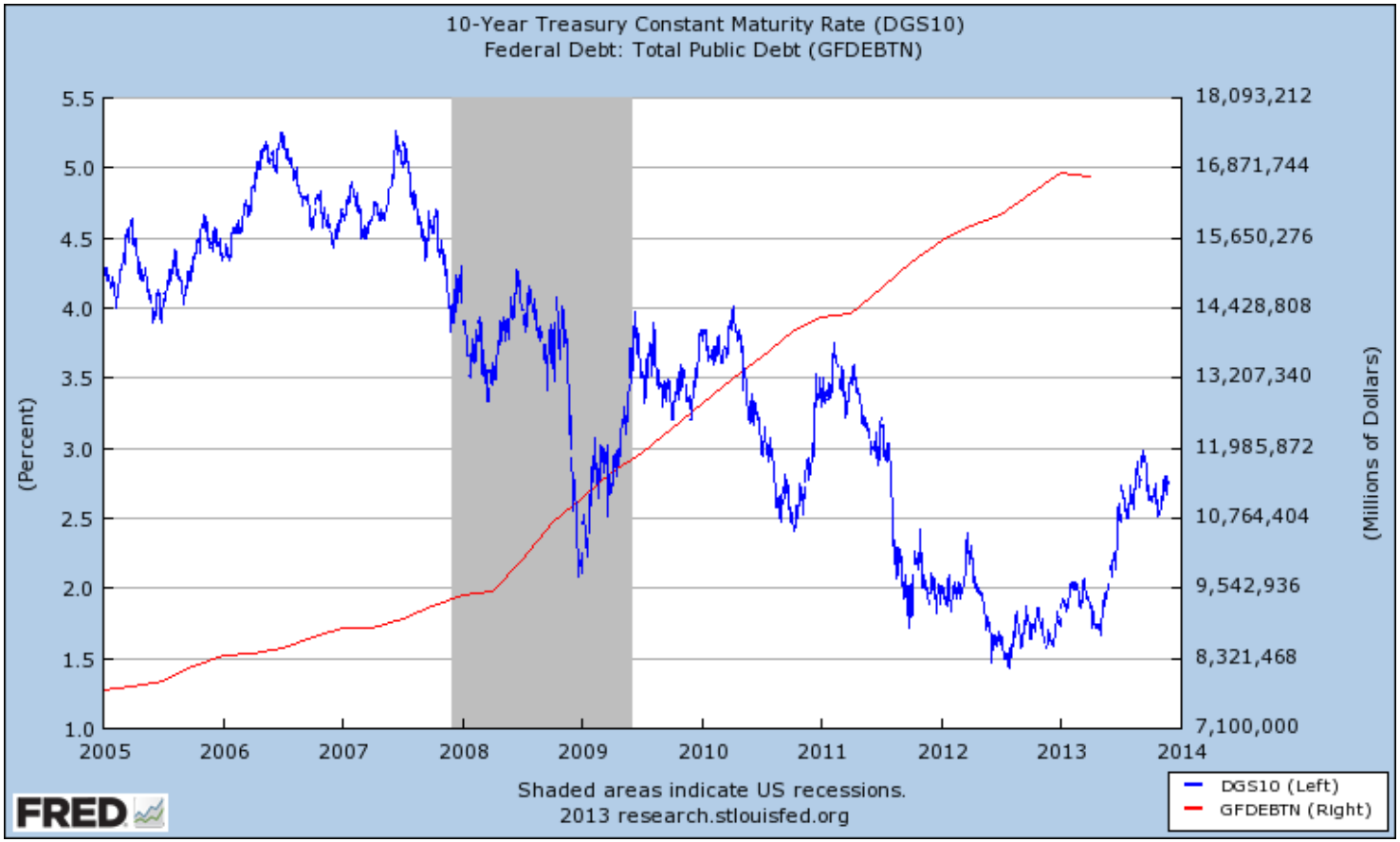


Figure: Government borrowing and interest rates.

Wrong prediction number 2

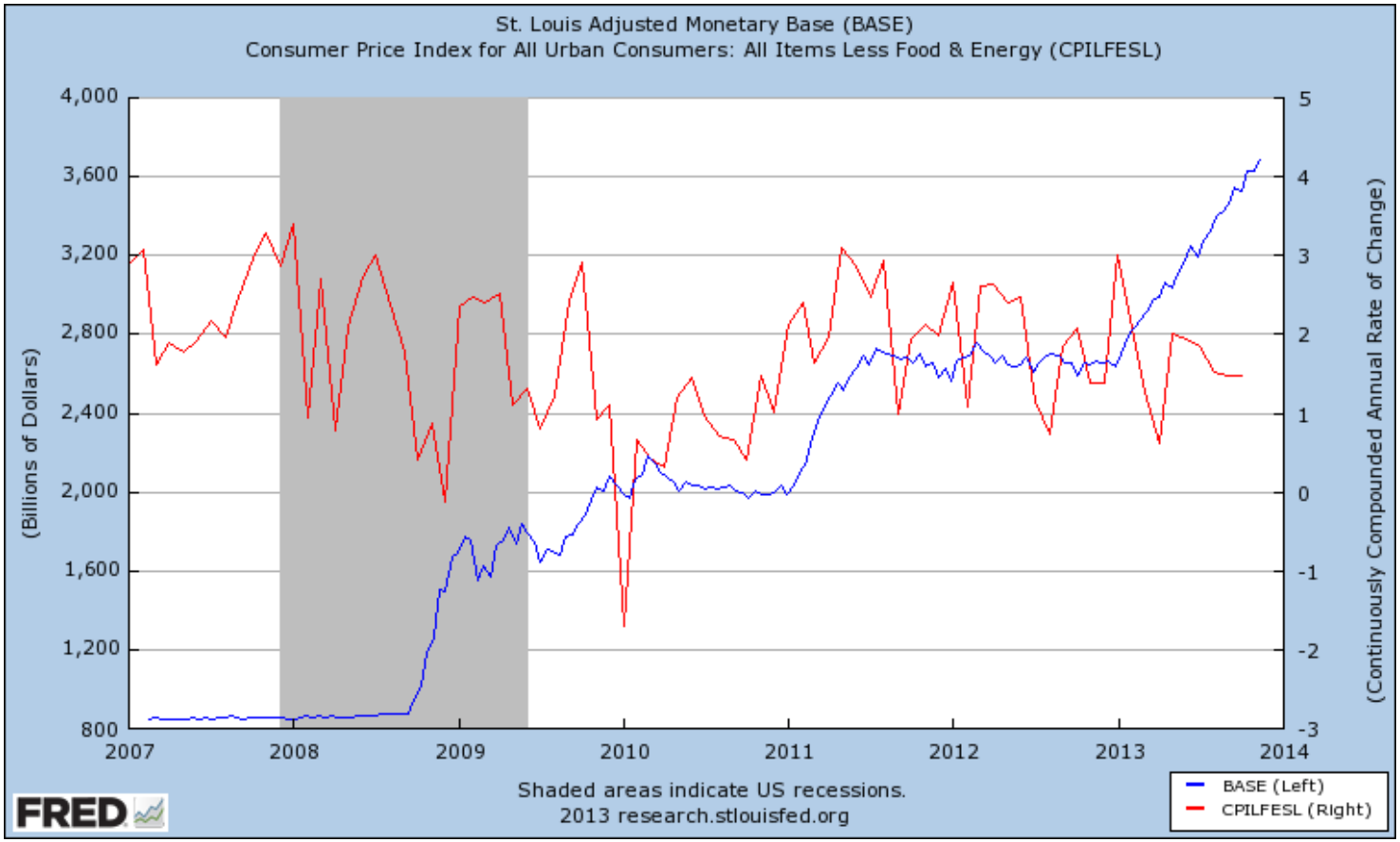


Figure: Monetary base and inflation.

Wrong prediction number 3

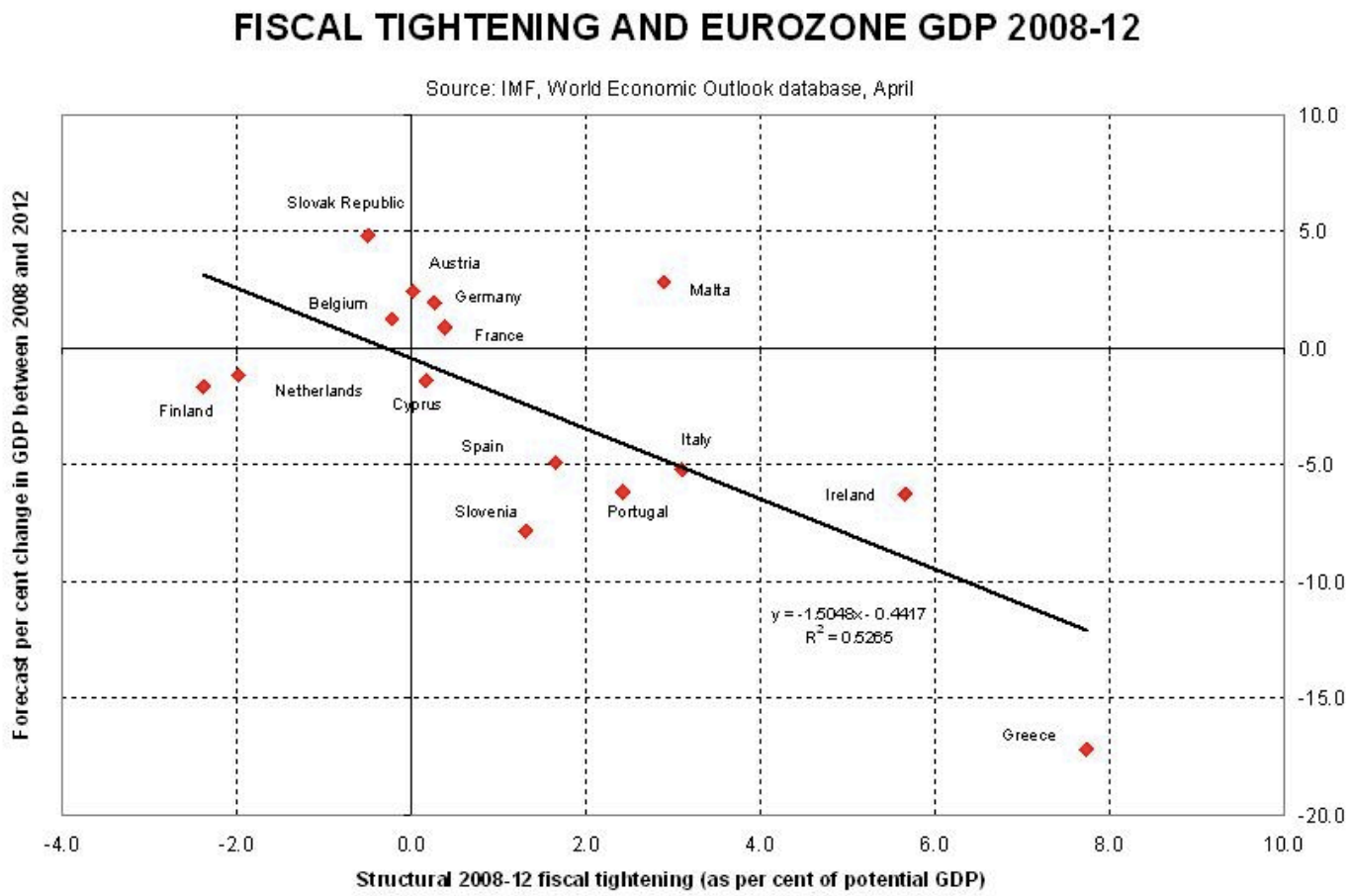


Figure: Fiscal tightening and GDP.

Better (but still bad) economics: soft core (saltwater) DSGE

- The strand of DSGE economists affiliated with New Keynesian theory got all these predictions right.
- They did so by augmented DSGE with 'imperfections' (wage stickiness, asymmetric information, imperfect competition, etc).
- Still DSGE at core - analogous to adding epicycles to Ptolemaic planetary system.
- For example: "Ignoring the foreign component, or looking at the world as a whole, the overall level of debt makes no difference to aggregate net worth – one person's liability is another person's asset." (Paul Krugman and Gauti B. Eggertsson, 2010, pp. 2-3)

Finance in DSGE models

- The financial sector merely serve as intermediaries channeling savings from households to business.
- Banks provide indirect finance by borrowing short and lending long (business loans), thereby solving the problem of liquidity preferences (Diamond and Dybvig (1986) model).
- Financial market provide direct finance through shares, thereby introducing market prices and discipline.
- Financial Frictions (e.g borrowing constraints, market liquidity) create persistence and amplification of real shocks (Bernanke and Gertler (1989), Kiyotaki and Moore (1997) models)
- See Brunnermeier and Sannikov (2013) for a recent contribution to this strand of literature in light of the financial crisis, in particular in the context of macro-prudential regulation.

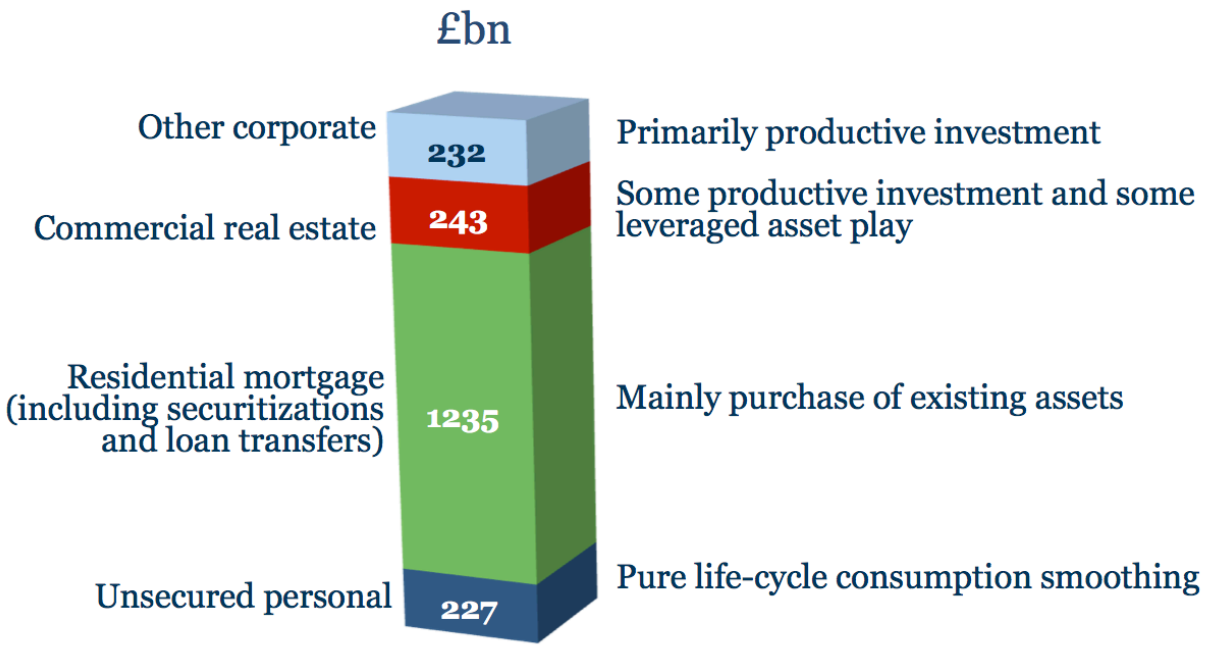
Frictions literature still missing the point

Turner 2013 observes that:

- "Quantitative impacts suggested by the models were far smaller than those empirically observed in real world episodes such as the Great Depression or the 2008 crisis"
- "Most of the literature omits consideration of behaviourally driven 'irrational' cycles in asset prices".
- "the vast majority of the literature ignores the possibilities of credit extension to finance the purchase of already existing assets".
- "the dominant model remains one in which household savers make deposits in banks, which lend money to entrepreneurs/businesses to pursue 'investment projects'. The reality of a world in which only a small proportion (e.g. 15%) of bank credit funds 'new investment projects' has therefore been left largely unexplored."

Turner (2013) slide

Categories of bank debt: UK, 2009

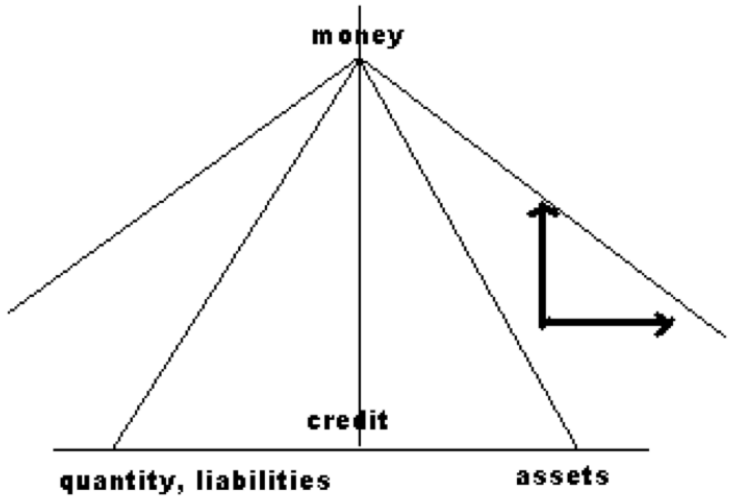


A parallel history of Macroeconomics

- Classical 19th century monetarism (Bagehot, Allan Young): role of banks in trade (Britain) and development (U.S.), central banking.
- Several prominent disciples of Keynes (Kaldor, Robinson, Davidson) immediately rejected the Neoclassical synthesis as "bastardized Keynesianism".
- Flow of Funds accounting - 1952 (Copeland): alternative to both $Y = C + I + G + X - M$ (finals sales) and $MV = PT$ (money transactions) by tracking exchanges of both goods and financial assets.
- Gurley, Shaw, Tobin, Minsky: financial intermediation at centre stage.
- Kindleberger (1978): detailed history of financial crises.
- Stock-flow consistent models (Godley, Lavoie)
- Revival of interest after the 2008 crisis.

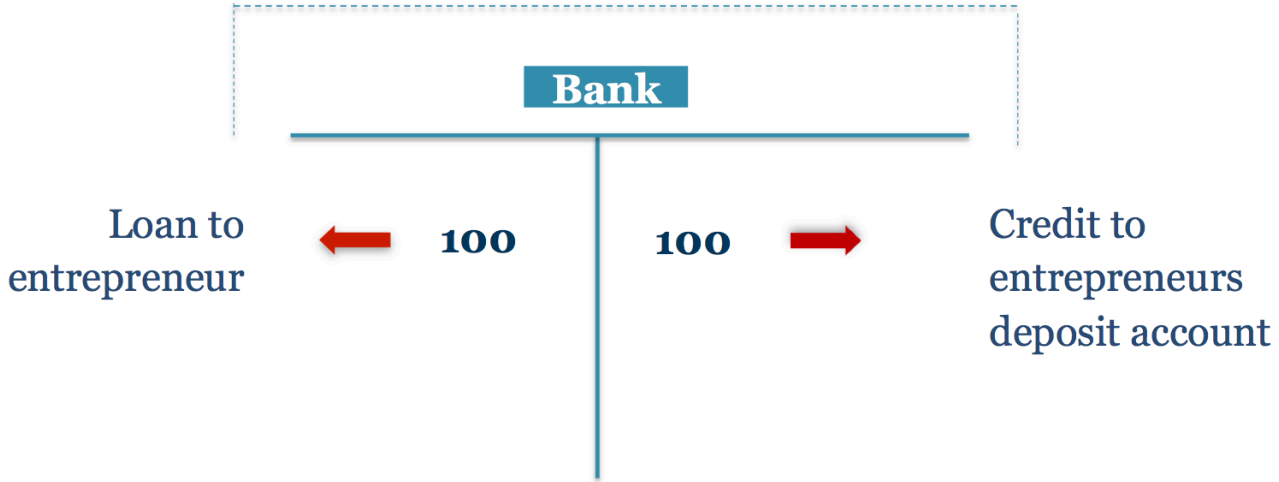
Key insight 1: money is not neutral

- Money is hierarchical: currency is a promise to pay gold (or taxes); deposits are promises to pay currency; securities are promises to pay deposits.
- Financial institutions are market-makers straddling two levels in the hierarchy: central banks, banks, security dealers.
- The hierarchy is dynamic: discipline and elasticity change in time.



Key insight 2: money is endogenous

- Banks create money and purchasing power.
- Reserve requirements are never binding.



Key insight 3: private debt matters

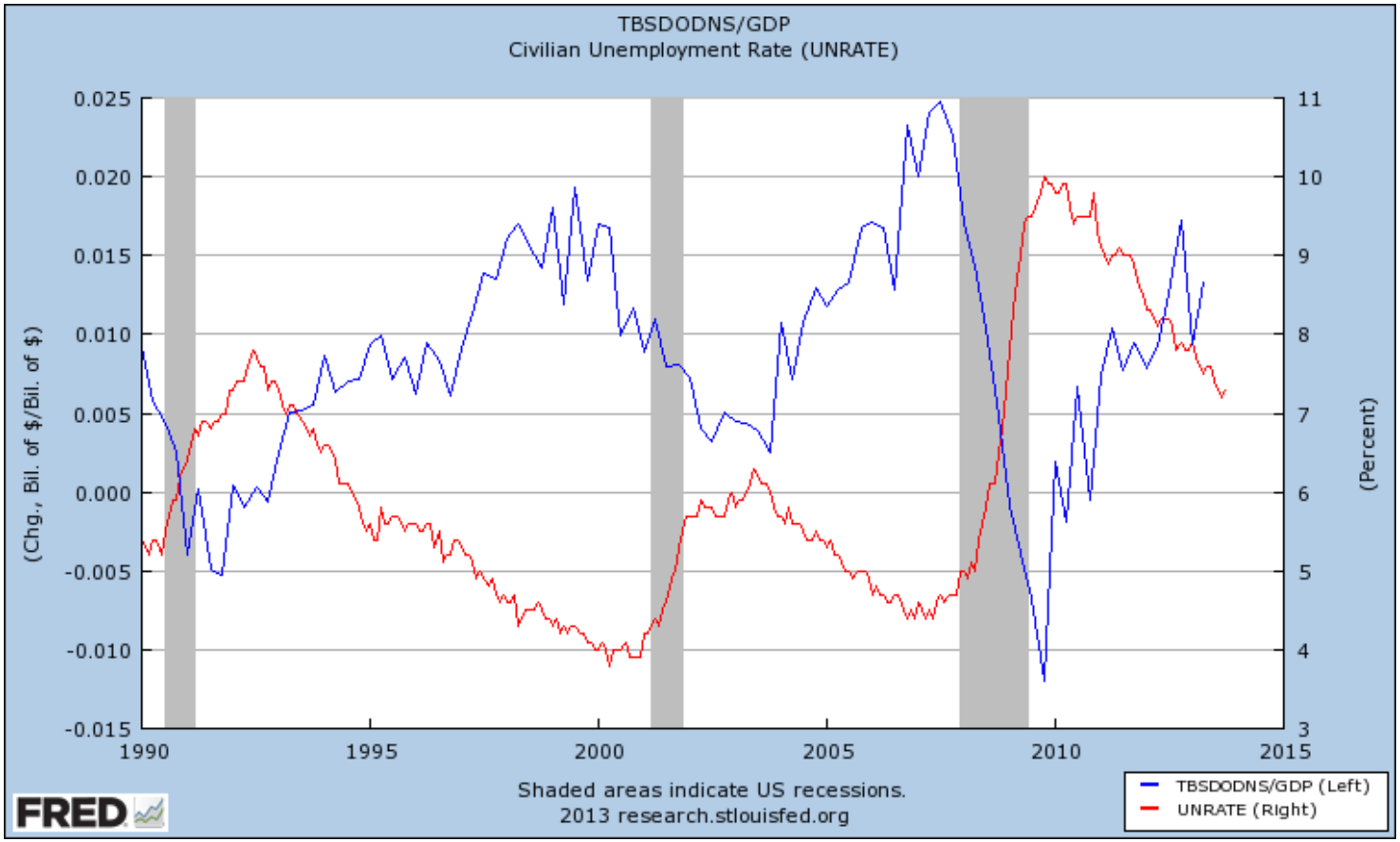


Figure: Change in debt and unemployment.

Key insight 4: finance is not just intermediation

- Market never clear in all states: set of events is larger than what can be contracted.
- The financial sector absorbs the risk of unfulfilled promises.
- The cone of acceptable losses defines the size of the real economy.

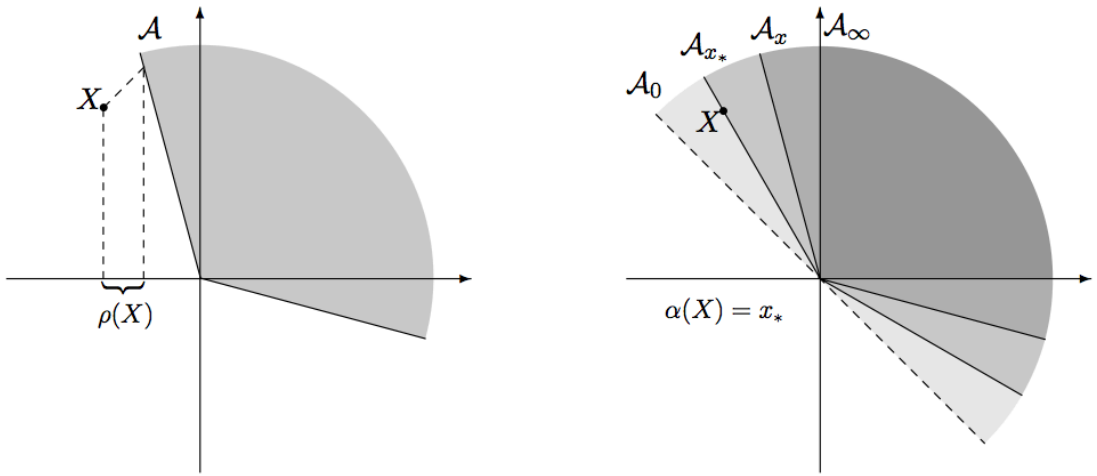


Figure: Cherny and Madan (2009)

Minsky's alternative interpretation of Keynes

- Neoclassical economics is based on barter paradigm: money is convenient to eliminate the double coincidence of wants.
- In a modern economy, firms make complex portfolios decisions: which assets to hold and how to fund them.
- Financial institutions determine the way funds are available for ownership of capital and production.
- Uncertainty in valuation of cash flows (assets) and credit risk (liabilities) drive fluctuations in real demand and investment.
- Economy is fundamentally cyclical, with each state (boom, crisis, deflation, stagnation, expansion and recovery) containing the elements leading to the next in an identifiable manner.

Minsky's Financial Instability Hypothesis

- Start when the economy is doing well but firms and banks are conservative.
- Most projects succeed - "Existing debt is easily validated: it pays to lever".
- Revised valuation of cash flows, exponential growth in credit, investment and asset prices.
- Highly liquid, low-yielding financial instruments are devalued, rise in corresponding interest rate.
- Beginning of "euphoric economy": increased debt to equity ratios, development of Ponzi financier.
- Viability of business activity is eventually compromised.
- Ponzi financiers have to sell assets, liquidity dries out, asset market is flooded.
- Euphoria becomes a panic.
- "Stability - or tranquility - in a world with a cyclical past and capitalist financial institutions is destabilizing".

Much better economics: SFC models

- Stock-flow consistent models emerged in the last decade as a common language for many heterodox schools of thought in economics.
- Consider both real and monetary factors from the start
- Specify the balance sheet and transactions between sectors
- Accommodate a number of behavioural assumptions in a way that is consistent with the underlying accounting structure.
- Reject silly (and mathematically unsound!) hypotheses such as the RARE individual (representative agent with rational expectations).
- See Godley and Lavoie (2007) for the full framework.

Balance Sheets

Balance Sheet	Households	Firms		Banks	Central Bank	Government	Sum
		current	capital				
Cash	$+H_h$			$+H_b$	$-H$		0
Deposits	$+M_h$		$+M_f$	$-M$			0
Loans			$-L$	$+L$			0
Bills	$+B_h$			$+B_b$	$+B_c$	$-B$	0
Equities	$+p_f E_f + p_b E_b$		$-p_f E_f$	$-p_b E_b$			0
Advances				$-A$	$+A$		0
Capital			$+pK$				pK
Sum (net worth)	V_h	0	V_f	V_b	0	$-B$	pK

Table: Balance sheet in an example of a general SFC model.

Transactions

Transactions	Households	Firms		Banks	Central Bank	Government	Sum
		current	capital				
Consumption	$-pC_h$	$+pC$		$-pC_b$			0
Investment		$+pI$	$-pI$				0
Gov spending		$+pG$				$-pG$	0
Acct memo [GDP]		$[pY]$					
Wages	$+W$	$-W$					0
Taxes	$-T_h$	$-T_f$				$+T$	0
Interest on deposits	$+r_M.M_h$	$+r_M.M_f$		$-r_M.M$			0
Interest on loans		$-r_L.L$		$+r_L.L$			0
Interest on bills	$+r_B.B_h$			$+r_B.B_b$	$+r_B.B_c$	$-r_B.B$	0
Profits	$+\Pi_d+\Pi_b$	$-\Pi$	$+\Pi_u$	$-\Pi_b$	$-\Pi_c$	$+\Pi_c$	0
Sum	S_h	0	S_f-pI	S_b	0	S_g	0

Table: Transactions in an example of a general SFC model.

Flow of Funds

Flow of Funds	Households	Firms		Banks	Central Bank	Government	Sum
		current	capital				
Cash	$+\dot{H}_h$			$+\dot{H}_b$	$-\dot{H}$		0
Deposits	$+\dot{M}_h$		$+\dot{M}_f$	$-\dot{M}$			0
Loans			$-\dot{L}$	$+\dot{L}$			0
Bills	$+\dot{B}_h$			$+\dot{B}_b$	$+\dot{B}_c$	$-\dot{B}$	0
Equities	$+p_f\dot{E}_f + p_b\dot{E}_b$		$-p_f\dot{E}_f$	$-p_b\dot{E}_b$			0
Advances				$-\dot{A}$	$+\dot{A}$		0
Capital			$+pI$				pI
Sum	S_h	0	S_f	S_b	0	S_g	pI
Change in Net Worth	$(S_h + \dot{p}_f E_f + \dot{p}_b E_b)$	$(S_f - \dot{p}_f E_f + \dot{p}K - p\delta K)$		$(S_b - \dot{p}_b E_b)$		S_g	$\dot{p}K + p\dot{K}$

Table: Flow of funds in an example of a general SFC model.

Agent-Based Models in Economics

- Agents have rational objectives, but realistic computational devices (inductive learning, bounded memory, limited information, war games, etc).
- Interactions are modelled directly, without fictitious clearing mechanisms.
- Hierarchical structures (i.e, banks are agents, but so are their clients, as well as the government).
- Equilibrium is just one possible outcome, not assumed a priori.
- Dynamic reactions can modify both existing interactions and the structure of the links.
- Mostly reliant on numerical simulations.

Mesoeconomics: an intermediate scale

- Heterogeneity is introduced through homogeneous 'types' or 'classes' of agents.
- Agents interact through mean field quantities computed over the classes.
- The state of the system is characterized by the number of agents in each class.
- Dynamics is modelled as a continuous-time Markov process.
- Key references are Foley (1994), Aoki (1996), Aoki and Yoshikawa 2006, Di Guilmi (2008) and Landini and Uberti (2008).

The Master Equation

- Let X_t be a Markov process with a discrete state space $\Gamma = \{x_k : 0 \leq k \leq N\}$ and define

$$P(x_k, t) := P[X_t = x_k | X_{t_0} = x], x \in \Gamma$$

- Recall that, for each x_k, the time evolution of the probability $P(x_k, t)$ is given by the so-called *master equation*

$$\frac{\partial P(x_k, t)}{\partial t} = \sum_{x_h \in \Gamma} \left[R_t(x_k | x_h) P(x_h, t) - R_t(x_h | x_k) P(x_k, t) \right],$$

where

$$R_t(x_k | x_h) := \lim_{\Delta_t \to 0^+} \frac{P[X_{t+\Delta} = x_k | X_t = x_h]}{\Delta t}$$

are the *transition probabilities*.

Birth-death processes

- The master equation simplifies considerably if $X_t = x_k$ is only allowed to jump to its nearest neighbours $x_{k\pm1}$ with transition probabilities $b(x_k, t)$ and $d(x_k, t)$, leading to

$$\begin{aligned}\frac{\partial P(x_k,t)}{\partial t} =& b(x_{k-1},t)P(x_{k-1},t) + d(x_{k+1},t)P(x_{k+1},t)\\ &- [b(x_k,t) + d(x_k,t)]P(x_k,t).\end{aligned}$$

- For example, these $N+1$ differential equations completely characterize the dynamics of an economy with N agents grouped into two types, where the (random) number of agents of each type at time t is given by the configuration vector $\mathbf{N}(t) = (X_t, N - X_t)$.

The homogenous master equation

- Continuing the previous example, let X_t be the number of agents of type 1 and write the master equation as

$$\frac{\partial P(x,t)}{\partial t} = (L-1)[d(x,t)P(x,t)]+(L^{-1}-1)[b(x,t)P(x,t)],$$

  where $L[a(x,t)] = a(x+1,t)$ is the lead operator.
- Observe further these operators can be written as

$$(L-1)a(x,t) = a(x+1,t) - a(x,t) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n a(x,t)}{\partial x^n}$$

$$(L^{-1}-1)a(x,t) = a(x-1,t) - a(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n a(x,t)}{\partial x^n}$$

The ansatz method

- Consider further the ansatz

$$X_t = Nm(t) + \sqrt{N}s_t,$$

where $m(t) = E[X_t]$ and s_t is a stochastic spread.

- If we then re-write $P(x,t) = Q(s,t)$, it follows that the probability density for the spread s satisfies the modified equation

$$\frac{\partial Q}{\partial t} - \sqrt{N}\frac{\partial Q}{\partial s}\frac{dm}{dt} = (L-1)[dQ] + (L^{-1}-1)[bQ],$$

where

$$b(s,t) = \lambda(t)(N - Nm - \sqrt{N}s), \quad d(s,t) = \mu(t)(Nm + \sqrt{N}s),$$

for given *transition rates* $\lambda(t)$ and $\mu(t)$.

Asymptotic approximation

- Rescaling the time variable as $\tau = t/N$ and collecting terms of order $N^{-1/2}$ and N^{-1} lead to the following system of couple equations

$$\frac{dm}{d\tau} = \lambda - (\lambda + \mu)m$$

$$\frac{\partial Q}{\partial \tau} = (\lambda + \mu)\frac{\partial}{\partial s}[sQ(s,\tau)] + \frac{\lambda(1-m) + \mu m}{2}\frac{\partial^2 Q(s,\tau)}{\partial s^2}$$

- In the special case when λ and μ are constant, we find the stationary solutions

$$\bar{m} = \frac{\lambda}{\mu + \lambda}$$

$$\bar{Q}(s) = C \exp\left(\frac{-s^2}{2\sigma^2}\right), \quad \sigma^2 = \frac{\lambda\mu}{(\lambda+\mu)^2}$$

A model with two types of firms - Carvalho and Di Guilmi (2015)

- Let $z = 1, 2$ denote speculative and hedge firms.
- Firm j invests according to

$$i^j_{z,t} = i_0 \cdot k^j_{z,t-1} + (\alpha_z \cdot \Pi + \beta) p \cdot q^j_{t-1} - \lambda \cdot b^j_{t-1},$$

  where $\alpha_z \geq 0, \beta \geq 0, \lambda \geq 0$ are known parameters, and $\alpha_1 > \alpha_2$, and Π is the profit share (see next page).
- The price for goods is assumed to be:

$$p = (1 + \tau)\frac{w}{\xi}$$

  where w is the nominal wage rate and ξ is productivity.
- The capital for firm j is then given by

$$k^j_t = (1 - \delta)k^j_{t-1} + i^j_t.$$

- Aggregate investment and capital are then

$$I_t = \sum_j i^j_{z,t}, \quad K_t = \sum_j k^j_t.$$

A model with two types of firms (continued)

- The wage and profit shares are given by

$$\Psi = \frac{wL}{pQ_t} = \frac{1}{1+\tau}, \quad \Pi = \frac{\tau}{1+\tau}.$$

- The household sector has disposable income of the form

$$Y_t = \Psi \cdot p \cdot Q_t + \Theta_t = \frac{1}{1+\tau} \cdot p \cdot Q_t + \sum_j \Theta_t^j,$$

  where Θ^j is distributed profits (see below).
- Household consumption is given by:

$$C_t = (1 - s_\Psi)\Psi \cdot p \cdot Q_t + (1-\sigma)W_{t-1}$$

  where σ is the propensity to save out of wealth.
- Output consists of consumption C_t plus investment I_t:

$$p \cdot Q_t = \frac{1+\tau}{\tau + s_\Psi}[I_t + (1-\sigma)W_{t-1}]$$

A model with two types of firms (continued)

- Total demand is allocated to each firm according to

$$q_t^j = Q_t \cdot \frac{k_t^j}{K_t} \cdot [1 + s_t^j],$$

where s^j is uniformly distributed with $E[s_t^j] = 0$.

- Firm j computes its retained profit by:

$$a_t^j = \Pi \cdot p \cdot q_t^j - r \cdot b_{t-1}^j = \frac{\tau}{1+\tau} \cdot p \cdot q_t^j - r \cdot b_{t-1}^j$$

- Update b_t^j:
  1. If $a_t^j < i_t^j + b_{t-1}^j$, then $\Delta b_t^j = i_t^j - a_t^j$
  2. If $a_t^j \geq i_t^j + b_{t-1}^j$, then $\Delta b_t^j = -b_{t-1}^j$ and the remainder goes to households with

$$\Theta_t^j = a_t^j - i_t^j - b_{t-1}^j \geq 0$$

A model with two types of firms (continued)

- Firm j bankrupts if:

$$\frac{b_t^j}{k_t^j} \geq c$$

  where c is positive constant.
- Household updates its wealth by:

$$\Delta W_t = S_t = Y_t - C_t$$

- Financial sector updates its net worth by:

$$\Delta \Omega_t^b = \bar{i} \cdot B_{t-1} = \bar{i} \cdot \sum_j b_{t-1}^j$$

Simulations versus ansatz solution - Carvalho and Di Guilmi (2015)

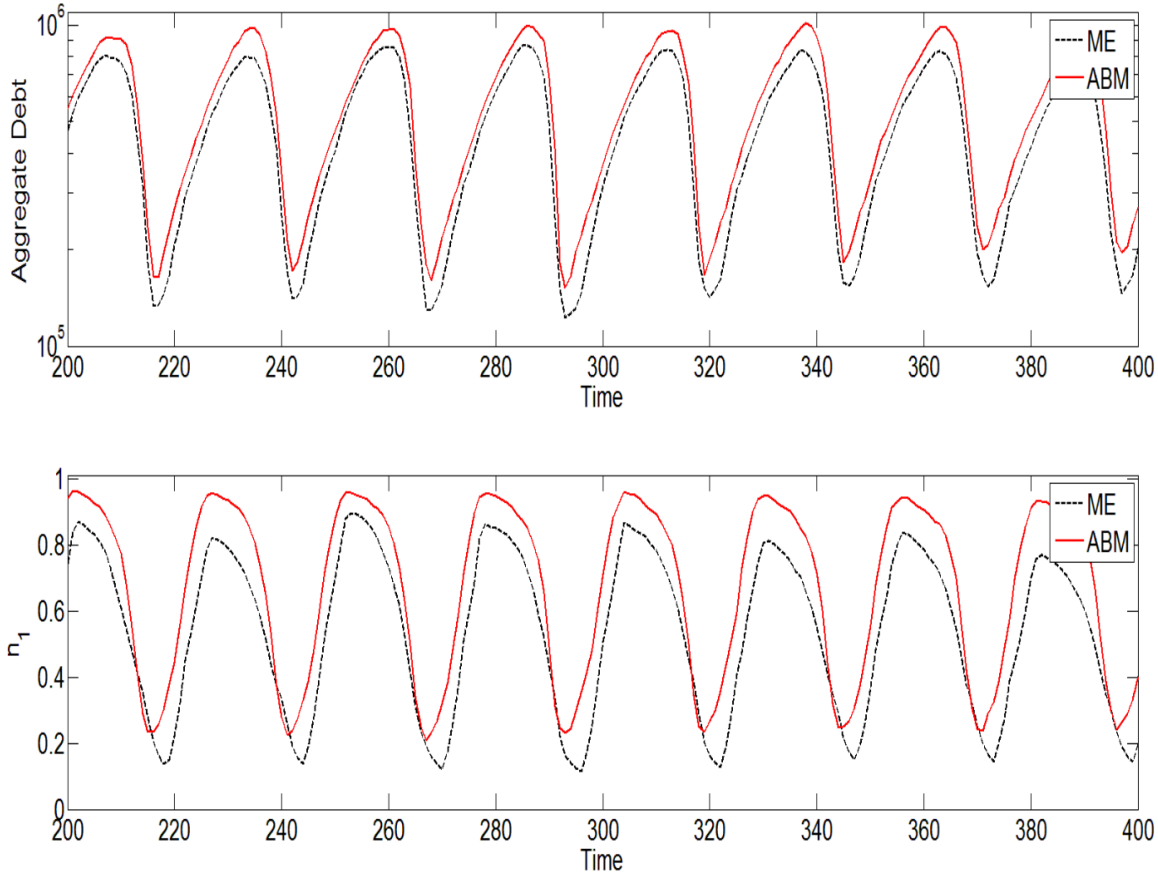


Figure 9: Comparison of the results of the agent-based model and the analytical solution for aggregate debt (upper panel) and share of speculative firms (lower panel).

Generalizations

- For the case of two types of agents in each of S sectors (e.g two types of firms, and two types of households, and two types of banks, etc), the ansatz method generalizes well and leads to S decoupled systems of two equations, each describing the mean and univariate distribution of the spread for N_1 in the pairs of occupation numbers of the form $(N_1^f, N^f - N_1^f)$, $(N_1^h, N^h - N_h^f)$, $(N_1^b, N^b - N_1^b)$, etc.
- The method fails for $k > 2$ types in each sector, as it gives a single system of two equations for the means of $N_1, N_2, \ldots, N_{k-1}$ and the joint of their spreads.
- Other solution methods, such as the van Kampen (1965) and the Kubo (1978) methods are available but have not been explored yet.

Two types of firms and two types of households

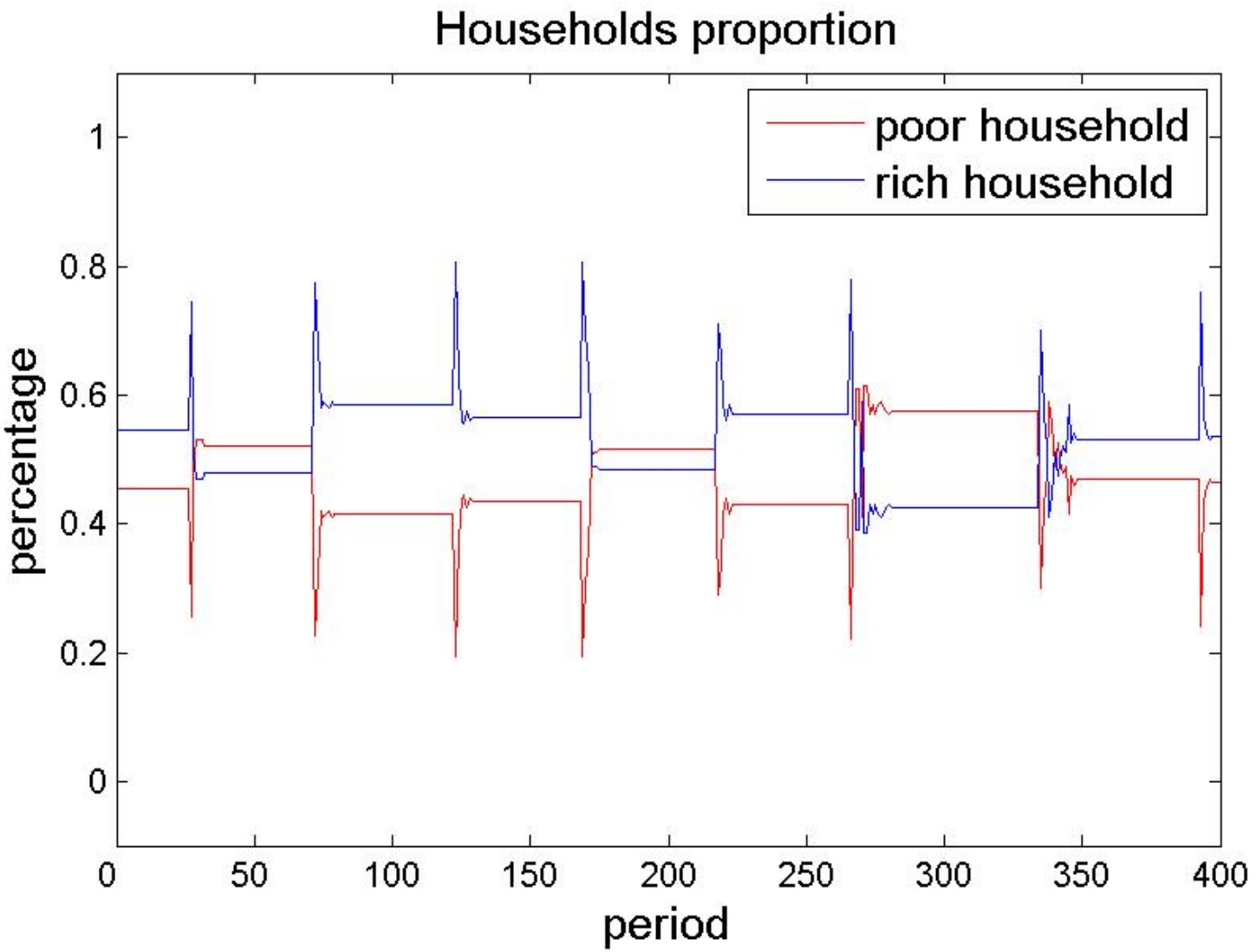
- Consider now the same model as before, but with two types of households characterized by their consumption

$$\begin{aligned} c^h_{1,t} &= (1-s_1)y^h_{t-1} + (1-\sigma)w^h_{t-1} \\ c^h_{2,t} &= (1-s_2)y^h_{t-1} + (1-\sigma)w^h_{t-1} \end{aligned}$$

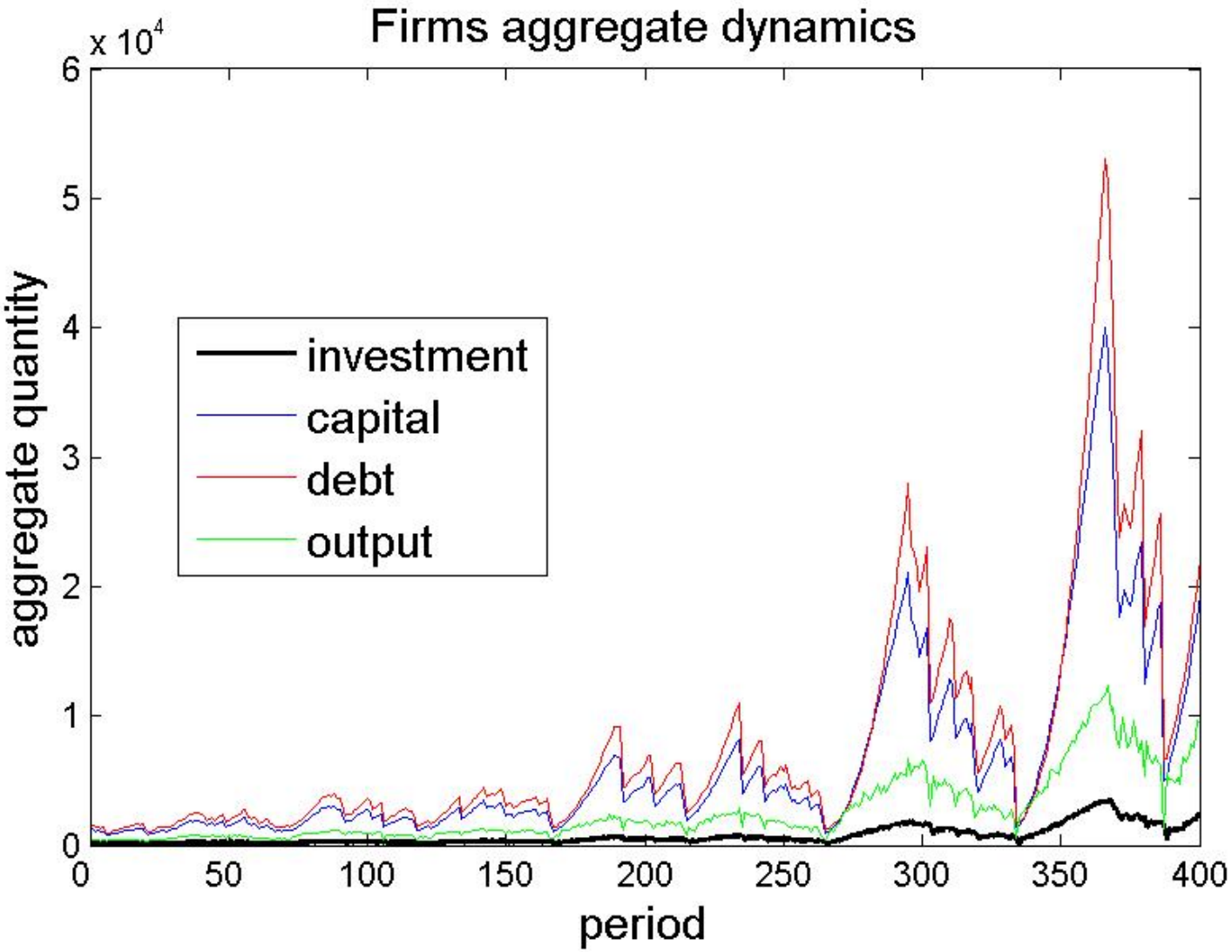
  where σ is the propensity to save out of wealth, s_1 and s_2 are the saving rate for household in type 1 and 2.
- Further assume $s_1 < s_2$, which implies household with lower income saves less than household with higher income.
- Household's saving is thus the difference between disposable income and consumption:

$$\begin{aligned} s^h_{1,t} &= y^h_{1,t} - c^h_{1,t} \\ s^h_{2,t} &= y^h_{2,t} - c^h_{2,t} \end{aligned}$$

Household proportions



Aggregate Dynamics



Concluding remarks

- Macroeconomics is too important to be left to macroeconomists.
- Since Keynes's death it has developed in two radically different approaches:
  1. The dominant one has the appearance of mathematical rigour (the SMD theorems notwithstanding), but is based on implausible assumptions, has poor fit to data in general, and is disastrously wrong during crises. Finance plays a negligible role
  2. The heterodox approach is grounded in history and institutional understanding, takes empirical work much more seriously, but is generally averse to mathematics. Finance plays a major role.
- It's clear which approach should be embraced by econophysics.

Thank you!