The priority option: the value of being a leader in complete and incomplete markets.

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leader
follower


follower
leader

Combining options and games

- Most of the real options approach consider monopolistic decision making.
- Option value leads to conservative exercise strategies.
- Intuitively, competition should erode the option value.
- A systematic application of both real options and game theory in strategic decisions has been proposed in the literature (see Smit and Trigeorgis (2004) for a review).
- The essential idea can be summarized in two rules:
  1. whenever the outcome of a given game involves a "wait–and–see" strategy, its pay-off should be calculated as the value of a real option;
  2. whenever the pay-off of a given involves a game, its value should calculated as the equilibrium solution to the game.
- In this way, option valuation and game theoretical equilibrium become dynamically related.

Competition in continuous times

- Consider the model of Grenadier (2000), where two firms contemplating the decision to pay a cost K to invest in a project leading to instantaneous cash flows $Y_t D_Q$ where

$$\frac{dY_t}{Y_t} = \nu dt + \eta dW_t, \tag{1}$$

where Y_t is a stochastic demand shock and D_Q is the inverse demand function when Q firms are present.

- Assume both market completeness and infinite maturity.
- More specifically, assume that Y_t is perfectly correlated with a traded financial asset

$$\frac{dP_t}{P_t} = \mu dt + \sigma dW_t = rdt + \sigma(dW_t + \lambda dt), \quad \lambda = \frac{\mu - r}{\sigma}. \tag{2}$$

Project value

- Let $\xi = \frac{\nu - r}{\eta}$ and $\lambda = \frac{\mu - r}{\sigma}$ be the Sharpe ratios for the project and the spanning asset.
- After both firms have invested, the value of the project is given by the expected value of all discounted future cash flows, that is

$$E^Q\left[\int_t^\infty e^{-r(s-t)} Y_s D_2 ds | Y_t = y\right] = \frac{yD_2}{\delta},$$

  where $\delta = \eta(\lambda - \xi)$.
- We see that δ plays the role of a dividend rate.
- Given that the leader has already invested, the value for the follower is then given by

$$F(y) = \sup_{\tau \geq 0} \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\Phi(Y_\tau)\mathbf{1}_{\{\tau<\infty\}} | Y_0 = y\right], \qquad (3)$$

  where τ is a stopping time and $\Phi(y) = D_2 y/\delta - K$.

Follower value

- The follower value then satisfies

$$\begin{cases} \frac{\eta^2}{2} y^2 F''(y) + (r-\delta) y F'(y) - rF(y) \leq 0 \\ F(y) \geq \Phi(y) \\ [F(y) - \Phi(y)] \left[\frac{\eta^2}{2} y^2 F''(y) + (r-\delta) y F'(y) - rF(y) \right] = 0. \end{cases} \quad (4)$$

supplemented by $F(v) \geq 0$ and $F(0) = 0$.

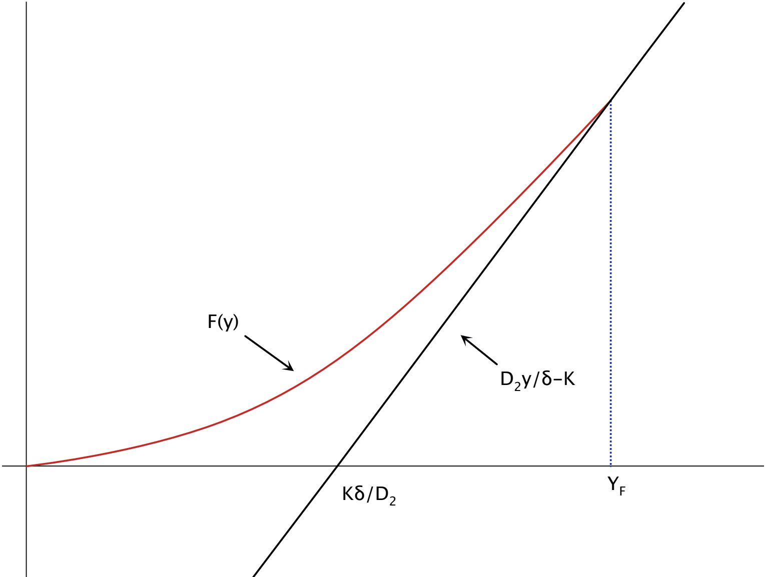
- The solution to variational inequality is

$$F(y) = \begin{cases} \frac{K}{\beta - 1} \left(\frac{y}{Y_F} \right)^{\beta}, & Y \leq Y_F \\ \frac{yD(2)}{\delta} - K, & y \geq Y_F \end{cases}$$

where $Y_F = \frac{\delta K \beta}{D_2(\beta - 1)}$ and $\beta > 1$ is a solution of

$$\frac{1}{2}\eta^2 \beta(\beta - 1) + (r - \delta)\beta = r.$$

Follower value



Leader value and simultaneous exercise

- After investing, the leader has no more options to exercise. As a result, the value of becoming a leader can be obtained entirely by expected value of future cash flow at a rate $Y_t D_1$ until the process Y reaches Y_F and $Y_t D_2$ thereafter.
- The solution to this simple first–passage–time problem is

$$L(y) = \begin{cases} \frac{yD(1)}{\delta} - \frac{D_1 - D_2}{D_2} \beta \frac{K}{\beta - 1} \left(\frac{y}{Y_F} \right)^{\beta} - K, & y < Y_F \\ \frac{yD_2}{\delta} - K, & y \geq Y_F \end{cases}$$

- Finally, it is clear that the value obtained from simultaneous exercise is

$$S(y) = \frac{yD_2}{\delta} - K$$

Threshold for the leader

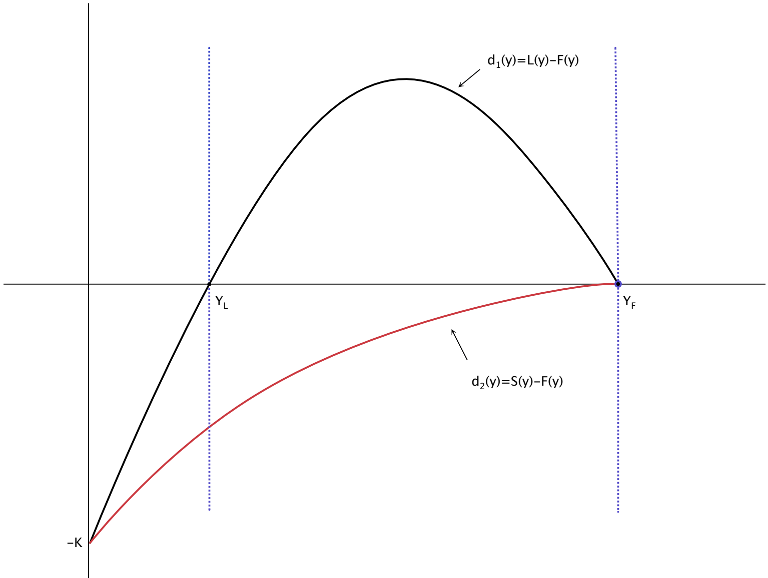
- It can be shown that there exists a unique point $Y_L \in (0, Y_F)$ such that

$$
\begin{aligned}
L(Y) &< F(Y), \quad Y < Y_L \\
L(Y) &= F(Y), \quad Y = Y_L \\
L(Y) &> F(Y), \quad Y_L < Y < Y_F \\
L(Y) &= F(Y), \quad Y \geq Y_F
\end{aligned}
$$

- In addition

$$
\begin{aligned}
S(Y) &< \min(L(Y), F(Y), \quad Y < Y_F \\
S(K) &= L(Y) = F(Y), \quad Y \geq Y_F
\end{aligned}
$$

Threshold for the leader



Randomized strategies

- Denote by $x_i(t) \in \{0,1\}$ the state for firm i at time t.
- A randomized strategy consists of a pair $(x_i(t), \mathcal{G}^i_t)$ where $\mathcal{G}^i$ is an independent enlargement of the market filtration $\mathcal{F}$.
- Denote $p_i(t) := P(x_i(t) = 1 | \mathcal{F}_t)$.
- A strategy is pure at time t if $p_i(t) \in \{0,1\}$.
- Otherwise it is mixed and represented by the randomization parameter $\eta_i(t)$ as

$$x_i(t) = 1_{\{\eta_i(t) \leq p_i(t)\}}, \quad \eta_i(t) \sim U[0,1], \quad \eta_i(t) \perp \mathcal{F}_t$$

- If $\mathcal{G}^1_t \cap \mathcal{G}^2_t = \mathcal{F}_t$, the strategies of the two players are independent.
- Alternatively, correlated equilibria can be introduced through a communication device $\gamma_{ij}(t)$ implemented via a third party.

Equilibrium strategies

- We focus on independent strategies only.
- Because of the Markovian structure of the market, it is enough to consider $p_i(t) \equiv p_i(Y_t)$.
- Assume that the game is played successively until one of the firms exercises.
- For $y \geq Y_F$ we have that $p^*(y) = p_1(y) = p_2(y) = 1$ is a Nash equilibrium.
- For $y \leq Y_L$ we have that $p^*(y) = p_1(y) = p_2(y) = 0$ is a Nash equilibrium.
- The interesting region is $Y_L < y < Y_F$.

Equilibrium strategies (continued)

- For $Y_L < y < Y_F$, the pay-off for firm i is

$$V_i = [p_i(1-p_j)L + p_i p_j S + (1-p_i)p_j F]\sum_{k=0}^{\infty}[(1-p_i)(1-p_j)]^k$$
$$= \frac{p_i(1-p_j)L + p_i p_j S + (1-p_i)p_j F}{1-(1-p_i)(1-p_j)}$$

- Maximizing this expression with respect to p_i and using symmetry leads to

$$p^*(y) = p_1(y) = p_2(y) = \frac{L(y)-F(y)}{L(y)-S(y)},$$

which can be shown to be a Nash equilibrium.

Expected payoff

- Observe that the expected payoff for each firm is

$$V(y) = \begin{cases} F(y), & y < Y_L \\ (1 - p_S)\frac{F(y)+L(y)}{2} + p_S S(y), & y \in (Y_L, Y_F) \\ S(y), & y > Y_F \end{cases},$$

  where $p_S(y)$ is the probability of simultaneous exercise.

- Using he expression for $p^*(y)$ we find

$$p_S(y) = \frac{p^*(y)^2}{1 - (1 - p^*(y))^2} = \frac{L - F}{L + F - 2S}$$

- This gives $V(y) = F(y)$ for all y !

Predetermined roles

- Define $L^\pi(Y)$ as the project value for a firm that has been predetermined as the Leader.
- Following the same reasoning as before, this value is given by

$$L^\pi(y) = \sup_{\tau \geq 0} \mathbb{E}^{\mathbb{Q}} \left[e^{-r\tau} L(Y_\tau) \mathbf{1}_{\{\tau < \infty\}} | Y_0 = y \right]. \qquad (5)$$

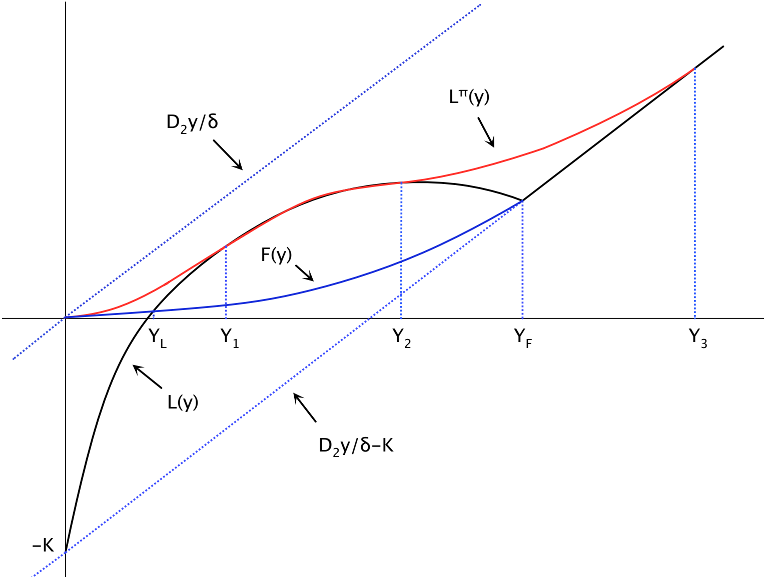
- Observe that

$$L'(y) = \begin{cases} \frac{D_1}{\delta} - \frac{(D_1 - D_2)\beta}{\delta} \left(\frac{y}{Y_F} \right)^{\beta - 1} & \text{if } y < Y_F \\ \frac{D_2}{\delta} & \text{if } y \geq Y_F \end{cases},$$

so $L(y)$ is not differentiable at Y_F.

- Moreover, $L''(y) < 0$ for $0 \leq y < Y_F$.

Obstacle problem for the leader



Variational inequality for the leader

- Formally, the value for a predetermined leader satisfies

$$\begin{cases} \frac{\eta^2}{2}y^2(L^\pi)'' + (r-\delta)y(L^\pi)' - rL^\pi \leq 0 \\ L^\pi(y) \geq L(y) \\ [L^\pi - L]\left[\frac{\eta^2}{2}y^2(L^\pi)'' + (r-\delta)y(L^\pi)' - rL^\pi\right] = 0, \end{cases} \tag{6}$$

supplemented by the conditions $L^\pi(y) \geq 0$ and $L(0) = 0$.

- Since the obstacle is not differentiable, there is not guarantee that this can be formulated in strong sense.

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Variational inequality in weak sense

- Define $M(y) = L(y) - \frac{D_2 y}{\delta} + K$, $\chi(y) = Ł(y) - \frac{D_2 y}{\delta} + K$ and $f(y) = rK - D_2 y$.
- Then the weak formulation of (**??**) is

$$b(M, \widetilde{M} - M) \geq (f, \widetilde{M} - M)_\kappa, \quad \forall \widetilde{M} \in \mathcal{K}, M \in \mathcal{K}.$$

- Here $b(\cdot, \cdot)$ is the bilinear form

$$\begin{aligned} b(g, \widetilde{g}) &= \int_0^\infty y g'(y) \left[\eta^2 \frac{1 - y^2(\kappa - 1)}{1 + y^2} - (r - \delta) \right] \widetilde{g}(y) \omega(y) dy \\ &+ \frac{1}{2} \int_0^\infty g'(y) \widetilde{g}'(y) y^2 \eta^2 \omega(y) dy + \int_0^\infty r g(y) \widetilde{g}(y) \omega(y) dy \end{aligned}$$

  on the Sobolev space $H^1_\kappa(0, \infty)$ and $(\cdot, \cdot)_\kappa$ is a weighted inner product on the Hilbert space $L^2_\kappa(0, \infty)$ with weight function $\omega(y) = 1/(1 + y^2)^\kappa$.
- Finally $\mathcal{K} = \{g \in H^1 | g \geq \chi, g(0) = K\}$.

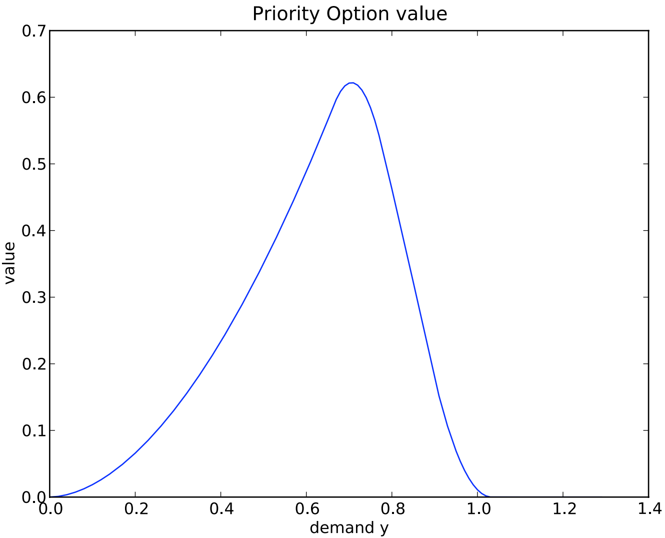
Leader value with priority

- Using a penalty approximation technique, we can show that M is smoother than the obstacle and satisfies a variational inequality in strong sense.
- It then follows that

$$L^{\pi}(y) = \begin{cases} Ay^{\beta} & \text{if } 0 \leq y < Y_1 \\ L(y) & \text{if } Y_1 \leq y \leq Y_2 \\ By^{\beta} + Cy^{\beta_1} & \text{if } Y_2 < y < Y_3 \\ \frac{D_2 y}{\delta} - K & \text{if } y \geq Y_3, \end{cases} \tag{7}$$

- Observe that $Y_L < Y_1$, so the priority option delays investment.
- The value of the priority option is then given by $\pi(y) = L^{\pi}(y) - F(y)$.

Priority option value



Incomplete markets

- Suppose now that the stochastic demand Y_t is correlated with the market portfolio P_t as follows:

$$\begin{cases} \frac{dY}{Y} = \nu dt + \eta dW_t \\ \frac{dP}{P} = \mu dt + \sigma dB_t \end{cases},$$

  where W_t and B_t have instantaneous correlation ρ.
- For simplicity, take $r = 0$.
- According to CAPM, if Y *could* be traded its equilibrium rate of return $\bar{\nu}$ would satisfy

$$\frac{\bar{\nu}}{\eta} = \rho \frac{\mu}{\sigma}$$

- We then define $\delta(\rho) := \bar{\nu} - \nu = \eta(\rho\lambda - \xi)$ as the below–equilibrium–shortfall–rate, which plays the role of a dividend yield in this case.

Utility problem

- As before, we calculate the project value when both firms have already invested as

$$E\left[\int_t^{\infty} e^{-\bar{\nu}(s-t)} Y_s D_2 ds \middle| Y_t = y\right] = \frac{yD_2}{\bar{\nu}-\nu} = \frac{yD_2}{\delta(\rho)}.$$

- For a utility function $U(x) = -e^{-\gamma x}$, define

$$F(x,y) = \sup_{(\tau,\theta)} \mathbb{E}\left[e^{\frac{\lambda^2 \tau}{2}} U\left(X_\tau^\theta + \left(\frac{D_2 Y_\tau}{\delta(\rho)} - K\right)^+\right)\right],$$

- Here $U(x) = -e^{-\gamma x}$ and

$$dX_t^\theta = \theta \frac{dP_t}{P_t} = \theta\sigma(\lambda dt + dW_t). \quad (8)$$

Follower value function

- Using Henderson (2007), let

$$\beta(\rho) = 1 + \frac{2\delta(\rho)}{\eta^2} > 1$$

and define $Y_F(\rho)$ as the solution to

$$\frac{D_2 Y_F(\rho)}{\delta(\rho)} - K = \frac{1}{\gamma(1-\rho^2)} \log\left[1 + \frac{\gamma(1-\rho^2) D_2 Y_F(\rho)}{\beta(\rho)\delta(\rho)}\right],$$

- Then

$$F(x,y) = \begin{cases} -e^{-\gamma x}\left[1 - \left(\frac{\gamma(1-\rho^2)D_2 Y_F}{\delta\beta + \gamma(1-\rho^2)D_2 Y_F}\right)\left(\frac{y}{Y_F}\right)^{\beta}\right]^{\frac{1}{1-\rho^2}}, & 0 \le y < Y_F \\ -e^{-\gamma x} e^{-\gamma\left(\frac{D_2 y}{\delta} - K\right)}, & y \ge Y_F \end{cases}$$

Leader value function

- As before, the value for the leader can be found by expected discounted cash–flows assuming that the follower exercises optimally:

$$L(x,y) = \begin{cases} -e^{-\gamma\left[x+\frac{D(1)}{\delta}y+\left(\frac{D(2)-D(1)}{\delta}\right)Y_F\left(\frac{y}{Y_F}\right)^{\alpha}-K\right]}, & 0\leq y\leq Y_F \\ -e^{-\gamma\left[x+\frac{D(2)}{\delta}y-K\right]}, & y\geq Y_F \end{cases},$$

where $\alpha = \left(\frac{1}{2}-\frac{\nu}{\eta^2}\right)+\sqrt{\left(\frac{1}{2}-\frac{\nu}{\eta^2}\right)^2+\frac{2\bar{\nu}}{\eta^2}}$

- Similarly, the value for simultaneous exercise is

$$S(x,y) = -e^{-\gamma\left[x+\frac{D(2)}{\delta}y-K\right]}$$

Leader threshold

- We can again show that, for each fixed x, there exists a unique point $Y_L \in (0, Y_F)$ such that

$$
\begin{aligned}
L(x,y) &< F(x,y), \quad y < Y_L \\
L(x,y) &= F(x,y), \quad y = Y_L \\
L(x,y) &> F(x,y), \quad Y_L < y < Y_F \\
L(x,y) &= F(x,y), \quad y \geq Y_F
\end{aligned}
$$

- In addition

$$
\begin{aligned}
S(x,y) &< \min(L(x,y), F(x,y), \quad y < Y_F \\
S(x,y) &= L(x,y) = F(x,y), \quad y \geq Y_F
\end{aligned}
$$

Equilibrium strategies

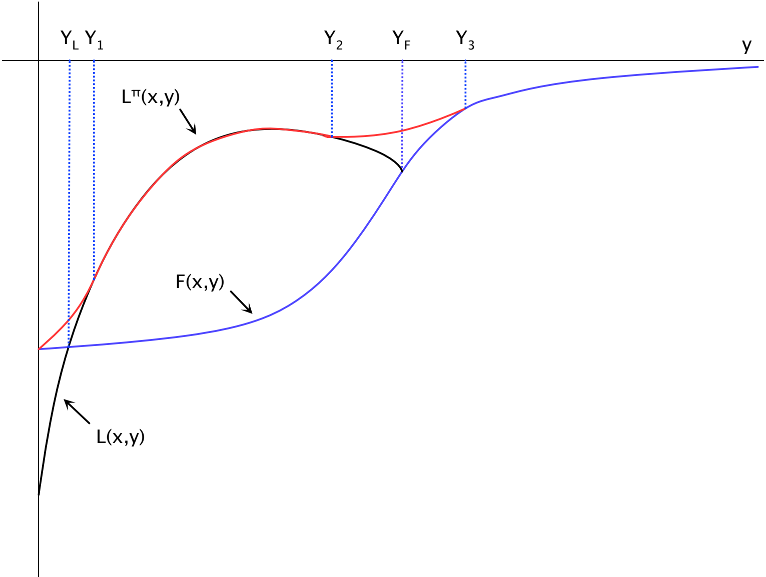
- Following the same arguments as before, we have that:
- For $y \geq Y_F$, $p^*(x,y) = p_1(x.y) = p_2(x,y) = 1$.
- For $y \leq Y_L$, $p^*(x,y) = p_1(x,y) = p_2(x,y) = 0$.
- For $Y_L < y < Y_F$.

$$p^*(x,y) = p_1(x,y) = p_2(x,y) = \frac{L(x,y) - F(x,y)}{L(x,y) - S(x,y)}.$$

The priority option

- Define $L^{\pi}(Y)$ as the expected utility for a firm that has been given a priority option for choosing to be the Leader.
- Formally, this has the same type of two-interval solution as in the complete market, but a rigorous proof is still open.
- The value for the priority option can then be obtained by an indifference value argument comparing $L^{\pi}(X, Y)$ and the equilibrium value V^{i} without the priority option.

Obstacle problem in incomplete markets



Conclusions

- Real options and game theory can be combined in a dynamic framework for decision making under uncertainty and competition.
- For a complete market, we found the leader and follower values as well as the equilibrium strategies for symmetric firms competing for an investment opportunity.
- Comparing this with the solution of a Stackelberg game gives the priority option value.
- The effects of incompleteness and risk aversion can be incorporated using the concept of indifference pricing.
- We again found the leader and follower values and equilibrium strategies.
- We characterize a candidate solution for the leader value with priority.
- Much more work is necessary for a large number of firms.
- Danke !