Macroeconomic modelling with heterogeneous agents: the master equation approach

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Alternative approaches

Mesoeconomic aggregation

Heterogeneous firms

Heterogeneous Households

Numerical Results

Macroeconomic modelling with heterogeneous agents: the master equation approach

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Mathematics and Statistics - McMaster University
Joint work with Patrick Li

Research **i**n **O**ptions , Rio de Janeiro, November 28, 2016

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A brief history of Macroeconomics

- Classics (Smith, Ricardo, Marx): no distinction between micro and macro, Say's law, emphasis on long run.
- Beginning of the 20th century (Wicksell, Fisher): natural rate of interest, quantity theory of money.
- Keynesian revolution (1936): shift to demand, fallacies of composition, role of expectations, and much more!
- Neoclassical synthesis - 1945 to 1970 (Hicks, Samuelson, Solow): Keynesian consensus.
- Rational Expectations Revolution - 1972 (Lucas, Prescott, Sargent): internal consistency, microfoundations.
- Start of Macro Wars: Real Business Cycles versus New Keynesian.
- 1990's: impression of consensus around DSGE models, but with different flavours.

Dynamic Stochastic General Equilibrium

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- Seeks to explain the aggregate economy using theories based on strong microeconomic foundations.
- Collective decisions of rational individuals over a range of variables for both present and future.
- All variables are assumed to be simultaneously in equilibrium.
- The only way the economy can be in disequilibrium at any point in time is through decisions based on wrong information.
- Money is neutral in its effect on real variables.

- The financial sector merely serve as intermediaries channeling savings from households to business.
- Banks provide indirect finance by borrowing short and lending long (business loans), thereby solving the problem of liquidity preferences (Diamond and Dybvig (1986) model).
- Financial market provide direct finance through shares, thereby introducing market prices and discipline.
- Financial Frictions (e.g borrowing constraints, market liquidity) create persistence and amplification of real shocks (Bernanke and Gertler (1989), Kiyotaki and Moore (1997) models)
- See Brunnermeier and Sannikov (2013) for a recent contribution to this strand of literature in light of the financial crisis, in particular in the context of macro-prudential regulation.

Frictions literature still missing the point

Turner 2013 observes that:

- "Quantitative impacts suggested by the models were far smaller than those empirically observed in real world episodes such as the Great Depression or the 2008 crisis"

- "Most of the literature omits consideration of behaviourally driven 'irrational' cycles in asset prices".

- "the vast majority of the literature ignores the possibilities of credit extension to finance the purchase of already existing assets".

- "the dominant model remains one in which household savers make deposits in banks, which lend money to entrepreneurs/businesses to pursue 'investment projects'. The reality of a world in which only a small proportion (e.g. 15%) of bank credit funds 'new investment projects' has therefore been left largely unexplored."

Categories of bank debt: UK, 2009

£bn

Other corporate — 232 — Primarily productive investment

Commercial real estate — 243 — Some productive investment and some leveraged asset play

Residential mortgage (including securitizations and loan transfers) — 1235 — Mainly purchase of existing assets

Unsecured personal — 227 — Pure life-cycle consumption smoothing
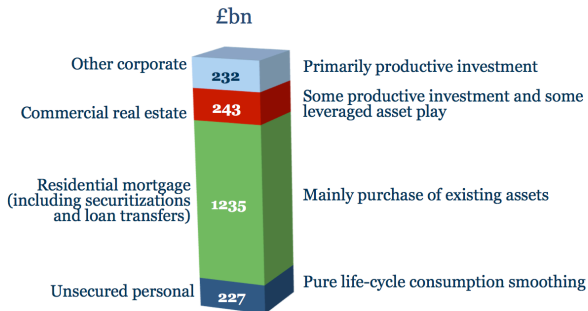
A parallel history of Macroeconomics

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- Classical 19th century monetarism (Bagehot, Allan Young): role of banks in trade (Britain) and development (U.S.), central banking.
- Several prominent disciples of Keynes (Kaldor, Robinson, Davidson) immediately rejected the Neoclassical synthesis as "bastardized Keynesianism".
- Flow of Funds accounting - 1952 (Copeland): alternative to both $Y = C + I + G + X - M$ (finals sales) and $MV = PT$ (money transactions) by tracking exchanges of both goods and financial assets.
- Gurley, Shaw, Tobin, Minsky: financial intermediation at centre stage.
- Kindleberger (1978): detailed history of financial crises.
- Stock-flow consistent models (Godley, Lavoie)
- Revival of interest after the 2008 crisis.

Key insight 1: money is not neutral

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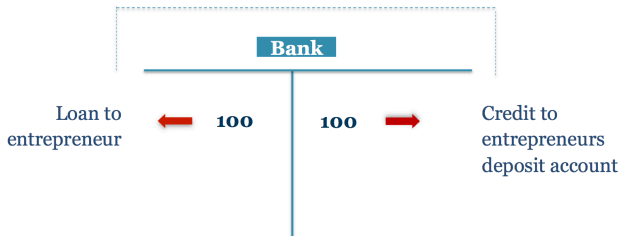
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- Money is hierarchical: currency is a promise to pay gold (or taxes); deposits are promises to pay currency; securities are promises to pay deposits.
- Financial institutions are market-makers straddling two levels in the hierarchy: central banks, banks, security dealers.
- The hierarchy is dynamic: discipline and elasticity change in time.
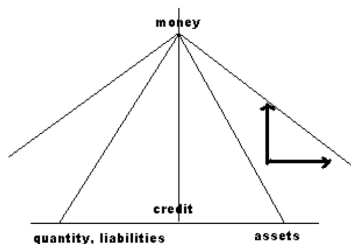
- Banks create money and purchasing power.
- Reserve requirements are never binding.
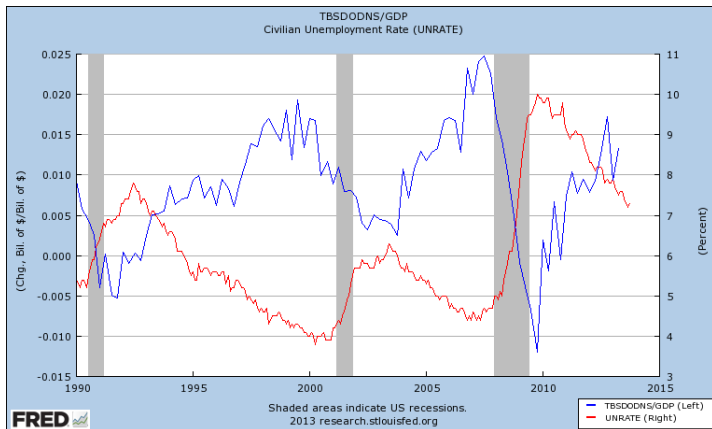
Key insight 3: private debt matters

Figure: Change in debt and unemployment.

Key insight 4: finance is not just intermediation

- Market never clear in all states: set of events is larger than what can be contracted.
- The financial sector absorbs the risk of unfulfilled promises.
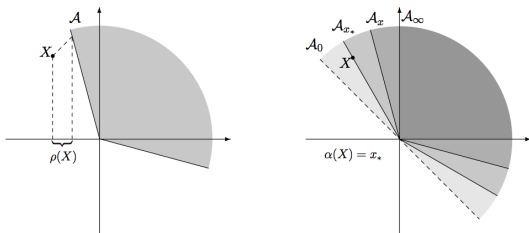- The cone of acceptable losses defines the size of the real economy.

Figure: Cherny and Madan (2009)

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Much better economics: SFC models

- Stock-flow consistent models emerged in the last decade as a common language for many heterodox schools of thought in economics.
- Consider both real and monetary factors from the start
- Specify the balance sheet and transactions between sectors
- Accommodate a number of behavioural assumptions in a way that is consistent with the underlying accounting structure.
- Reject silly (and mathematically unsound!) hypotheses such as the RARE individual (representative agent with rational expectations).
- See Godley and Lavoie (2007) for the full framework.

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Balance Sheets

Balance Sheet	Households	Firms current	Firms capital	Banks	Central Bank	Government	Sum
Cash	$+H_h$			$+H_b$	$-H$		0
Deposits	$+M_h$		$+M_f$	$-M$			0
Loans			$-L$	$+L$			0
Bills	$+B_h$			$+B_b$	$+B_c$	$-B$	0
Equities	$+p_f E_f + p_b E_b$		$-p_f E_f$	$-p_b E_b$			0
Advances				$-A$	$+A$		0
Capital			$+pK$				pK
Sum (net worth)	V_h	0	V_f	V_b	0	$-B$	pK

Table: Balance sheet in an example of a general SFC model.

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Transactions

Transactions	Households	Firms		Banks	Central Bank	Government	Sum
		current	capital				
Consumption	$-pC_h$	$+pC$		$-pC_b$			0
Investment		$+pI$	$-pI$				0
Gov spending		$+pG$				$-pG$	0
Acct memo [GDP]		$[pY]$					
Wages	$+W$	$-W$					0
Taxes	$-T_h$	$-T_f$				$+T$	0
Interest on deposits	$+r_M.M_h$	$+r_M.M_f$		$-r_M.M$			0
Interest on loans		$-r_L.L$		$+r_L.L$			0
Interest on bills	$+r_B.B_h$			$+r_B.B_b$	$+r_B.B_c$	$-r_B.B$	0
Profits	$+\Pi_d+\Pi_b$	$-\Pi$	$+\Pi_u$	$-\Pi_b$	$-\Pi_c$	$+\Pi_c$	0
Sum	S_h	0	S_f-pI	S_b	0	S_g	0

Table: Transactions in an example of a general SFC model.

Flow of Funds	Households	Firms		Banks	Central Bank	Government	Sum
		current	capital				
Cash	$+\dot{H}_h$			$+\dot{H}_b$	$-\dot{H}$		0
Deposits	$+\dot{M}_h$		$+\dot{M}_f$	$-\dot{M}$			0
Loans			$-\dot{L}$	$+\dot{L}$			0
Bills	$+\dot{B}_h$			$+\dot{B}_b$	$+\dot{B}_c$	$-\dot{B}$	0
Equities	$+p_f\dot{E}_f + p_b\dot{E}_b$		$-p_f\dot{E}_f$	$-p_b\dot{E}_b$			0
Advances				$-\dot{A}$	$+\dot{A}$		0
Capital			$+pI$				pI
Sum	S_h	0	S_f	S_b	0	S_g	pI
Change in Net Worth	$(S_h + \dot{p}_f E_f + \dot{p}_b E_b)$		$(S_f - \dot{p}_f E_f + \dot{p}K - p\delta K)$	$(S_b - \dot{p}_b E_b)$		S_g	$\dot{p}K + p\dot{K}$

Table: Flow of funds in an example of a general SFC model.

Agent-Based Models in Economics

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- Agents have rational objectives, but realistic computational devices (inductive learning, bounded memory, limited information, war games, etc).

- Interactions are modelled directly, without fictitious clearing mechanisms.

- Hierarchical structures (i.e, banks are agents, but so are their clients, as well as the government).

- Equilibrium is just one possible outcome, not assumed a priori.

- Dynamic reactions can modify both existing interactions and the structure of the links.

- Mostly reliant on numerical simulations.

- Heterogeneity is introduced through homogeneous 'types' or 'classes' of agents.
- Agents interact through mean field quantities computed over the classes.
- The state of the system is characterized by the number of agents in each class.
- Dynamics is modelled as a continuous-time Markov process.
- Key references are Foley (1994), Aoki (1996), Aoki and Yoshikawa 2006, Di Guilmi (2008) and Landini and Uberti (2008).

The Master Equation

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- Let X_t be a Markov process with a discrete state space $\Gamma = \{x_k : 0 \leq k \leq N\}$ and define

$$P(x_k, t) := P[X_t = x_k | X_{t_0} = x], x \in \Gamma$$

- Recall that, for each x_k, the time evolution of the probability $P(x_k, t)$ is given by the so-called *master equation*

$$\frac{\partial P(x_k, t)}{\partial t} = \sum_{x_h \in \Gamma} \left[R_t(x_k | x_h) P(x_h, t) - R_t(x_h | x_k) P(x_k, t) \right],$$

where

$$R_t(x_k | x_h) := \lim_{\Delta_t \to 0^+} \frac{P[X_{t+\Delta} = x_k | X_t = x_h]}{\Delta t}$$

are the *transition probabilities*.

Birth-death processes

- The master equation simplifies considerably if $X_t = x_k$ is only allowed to jump to its nearest neighbours $x_{k\pm1}$ with transition probabilities $b(x_k)$ and $d(x_k)$, leading to

$$\frac{\partial P(x_k, t)}{\partial t} = b(x_{k-1})P(x_{k-1}, t) + d(x_{k+1})P(x_{k+1}, t)$$
$$- [b(x_k) + d(x_k)]P(x_k, t).$$

- For example, these $N + 1$ differential equations completely characterize the dynamics of an economy with N agents grouped into two types, where the (random) number of agents of each type at time t is given by the configuration vector $\mathbf{N}(t) = (X_t, N - X_t)$.

The homogenous master equation

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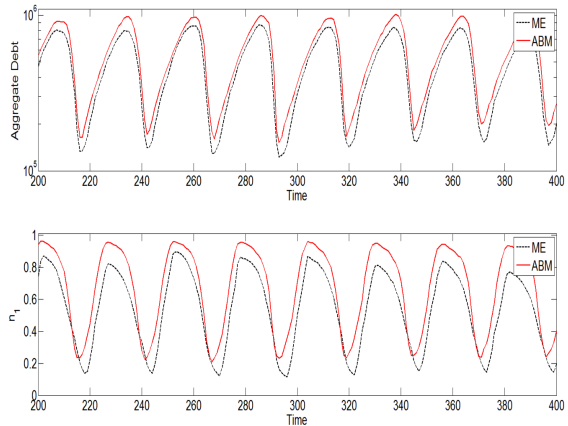Households

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- Continuing the previous example, let X_t be the number of agents of type 1 and write the master equation as

$$\frac{\partial P(x,t)}{\partial t} = (L-1)[d(x)P(x,t)] + (L^{-1}-1)[b(x)P(x,t)],$$

where $L[a(x,t)] = a(x+1,t)$ is the lead operator.

- Observe further these operators can be written as

$$(L-1)a(x,t) = a(x+1,t) - a(x,t) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n a(x,t)}{\partial x^n}$$

$$(L^{-1}-1)a(x,t) = a(x-1,t) - a(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n a(x,t)}{\partial x^n}$$

- Consider further the ansatz

$$X_t = Nm(t) + \sqrt{N}s_t,$$

where $m(t) = E[X_t]$ and s_t is a stochastic spread.

- If we then re-write $P(x, t) = Q(s, t)$, it follows that the probability density for the spread s satisfies the modified equation

$$\frac{\partial Q}{\partial t} - \sqrt{N}\frac{\partial Q}{\partial s}\frac{dm}{dt} = (L - 1)[dQ] + (L^{-1} - 1)[bQ],$$

where

$$b(s) = \lambda(N - Nm - \sqrt{N}s), \quad d(s) = \mu(Nm + \sqrt{N}s),$$

for given *individual* transition rates λ and μ.

Asymptotic approximation

- Collecting terms of order $N^{-1/2}$ and N^{-1} lead to the following system of couple equations

$$\frac{dm}{d\tau} = \lambda - (\lambda + \mu)m$$

$$\frac{\partial Q}{\partial \tau} = (\lambda + \mu)\frac{\partial}{\partial s}[sQ(s,\tau)] + \frac{\lambda(1-m) + \mu m}{2}\frac{\partial^2 Q(s,\tau)}{\partial s^2}$$

- These lead to stationary solution of the form

$$\bar{m} = \frac{\lambda}{\mu + \lambda}$$

$$\bar{Q}(s) = C\exp\left(\frac{-s^2}{2\sigma^2}\right), \quad \sigma^2 = \frac{\lambda\mu}{(\lambda+\mu)^2}$$

A model with two types of firms - inspired by Carvalho and Di Guilmi (2015)

- Let $z = 1, 2$ denote aggressive and conservative firms with investment for firm j given by

$$i_{z,t}^{j} = (\alpha_z \cdot \pi + \beta)p \cdot q_{t-1}^{j} - \lambda \cdot b_{t-1}^{j},$$

where $\alpha_z \geq 0, \beta \geq 0, \lambda \geq 0$ are known parameters, and $\alpha_1 > \alpha_2$, and π is the profit share (see next page).

- The price for goods is assumed to be:

$$p = \psi^{-1} c = \psi^{-1} \frac{w}{\xi},$$

where ψ^{-1} is a mark-up factor over unit labour cost c, w is the nominal wage rate, and ξ is productivity per worker.

- The capital for firm j is then given by

$$k_t^{j} = (1 - \delta)k_{t-1}^{j} + i_t^{j}.$$

leading to $I_t = \sum_j i_{z,t}^{j}$ and $K_t = \sum_j k_t^{j}$

A model with two types of firms (continued)

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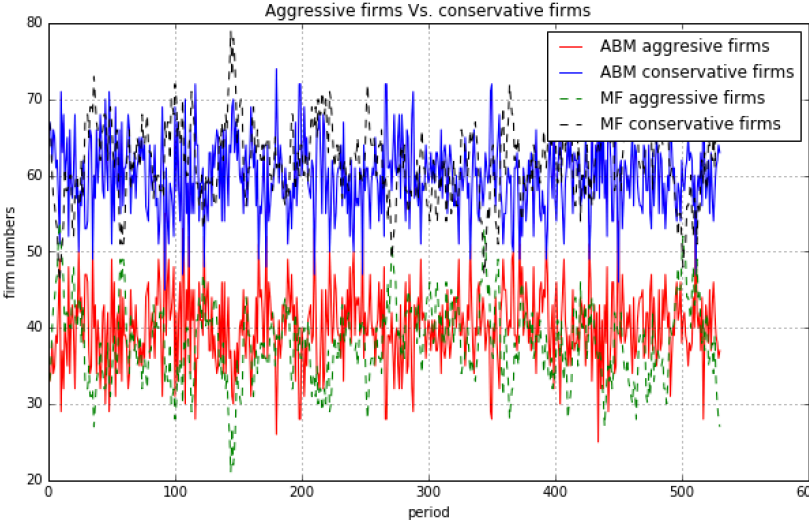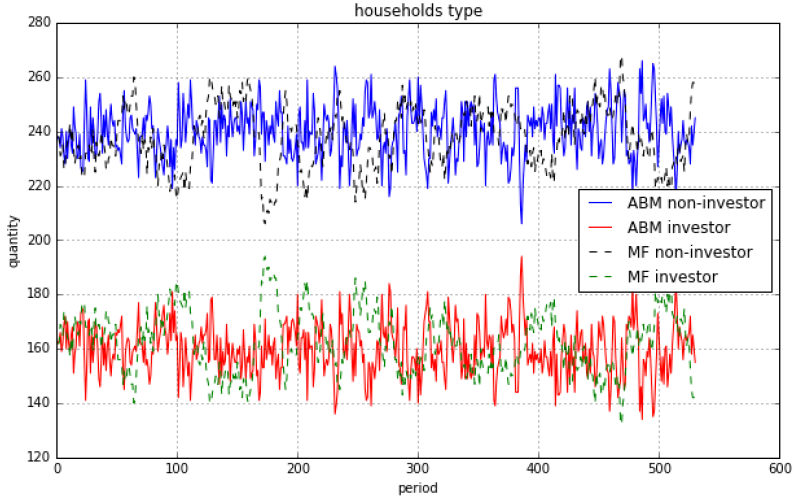heterogeneous
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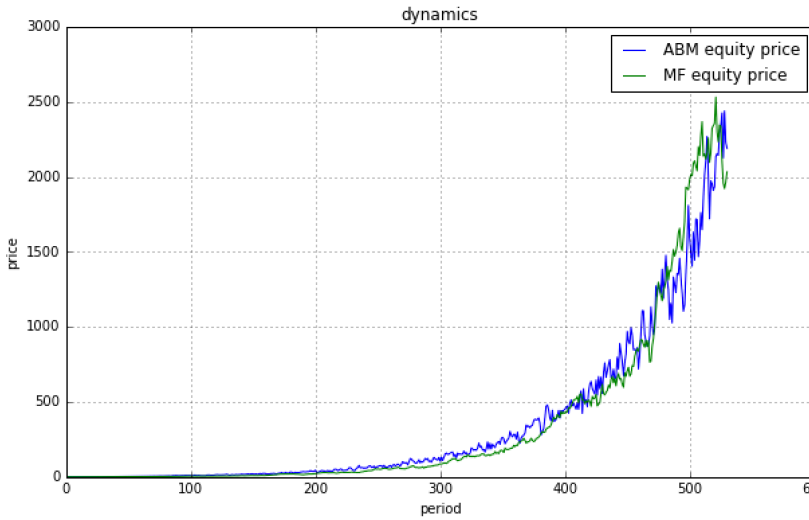Heterogeneous
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- The wage and profit shares are given by

$$\frac{w\ell}{p\xi\ell} = \psi, \quad \Pi = 1 - \psi.$$

- The household sector has disposable income of the form

$$Y_t = \psi \cdot p \cdot Q_t + \theta_t,$$

where $\theta_t = \sum_j \theta_t^j$ correspond to distributed profits (see below).

- Household consumption is given by:

$$C_t = (1 - s^y)Y_t + (1 - s^v)V_{t-1}$$

- Output consists of consumption C_t plus investment I_t:

$$p \cdot Q_t = \frac{I_t + (1 - s^y)\theta_t + (1 - s^v)V_{t-1}}{1 - \psi(1 - s^y)}$$

- Total demand is allocated to each firm according to

$$q_t^j = Q_t \cdot \frac{k_t^j}{K_t}.$$

- Firm j computes its retained profit as:

$$a_t^j = (1 - \Theta)(\pi \cdot p \cdot q_t^j - r \cdot b_{t-1}^j)$$

- This leads to a change in debt of the form:

$$\Delta b^j = i_t^j - a_t^j$$

- The wealth of the household sector, which consists entirely of deposits, changes according to

$$\Delta V_t = Y_t - C_t$$

Figure 9: Comparison of the results of the agent-based model and the analytical solution for aggregate debt (upper panel) and share of speculative firms (lower panel).

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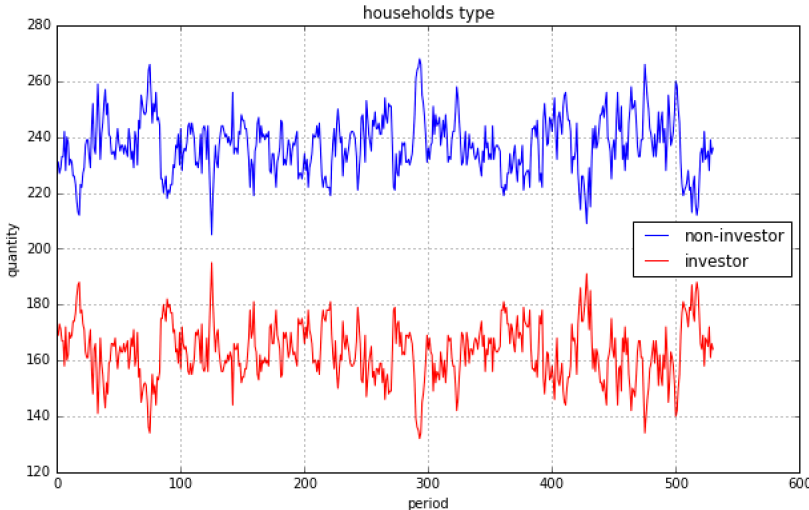agents: the
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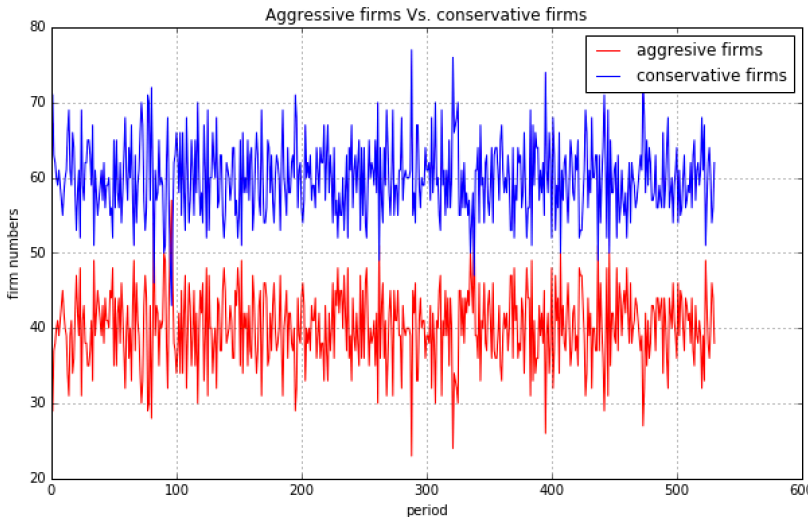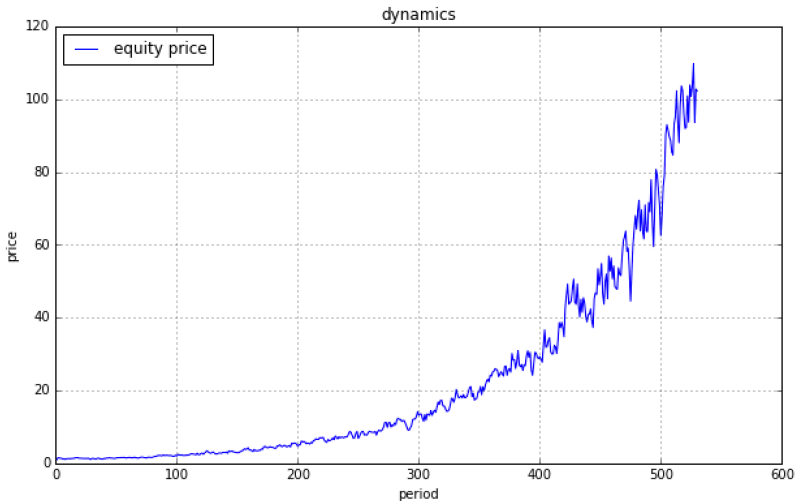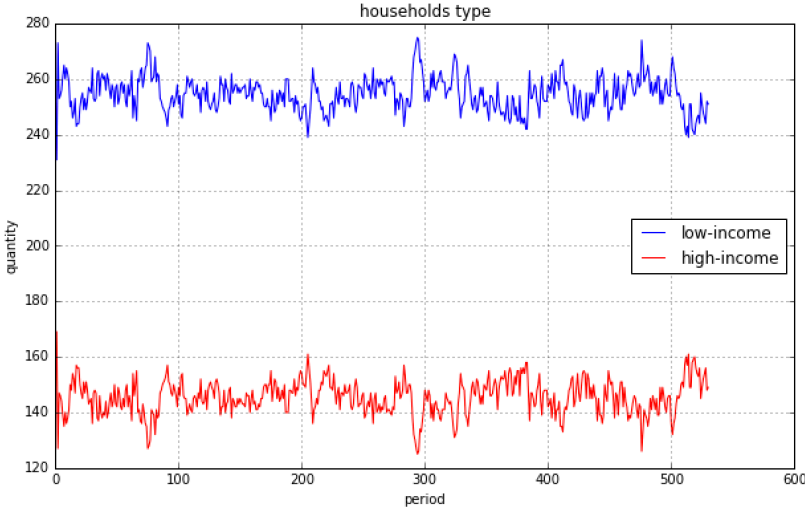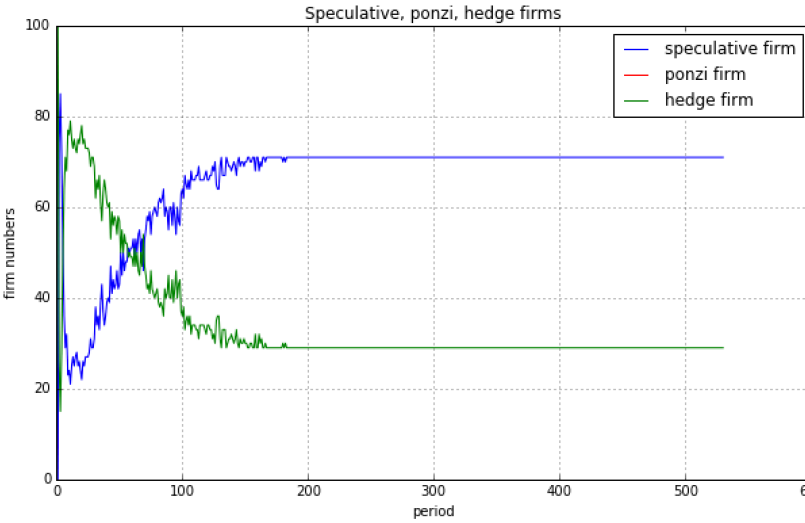Alternative
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Generalizations

- For the case of two types of agents in each of S sectors (e.g two types of firms, and two types of households, and two types of banks, etc), the ansatz method generalizes well and leads to S decoupled systems of two equations, each describing the mean and univariate distribution of the spread for N_1 in the pairs of occupation numbers of the form $(N_1^f, N^f - N_1^f)$, $(N_1^h, N^h - N_1^f)$, $(N_1^b, N^b - N_1^b)$, etc.

- The method fails for $k > 2$ types in each sector, as it gives a single system of two equations for the means of $N_1, N_2, \ldots, N_{k-1}$ and the joint of their spreads.

- Other solution methods, such as the van Kampen (1965) and the Kubo (1978) methods are available but have not been explored yet.

- Consider now the same model as before, but with two types of households, workers and investors, characterized by their consumption

$$
\begin{aligned}
c_{1,t}^h &= (1 - s_1^y)y_{t-1}^h + (1 - s_1^v)w_{t-1}^h \\
c_{2,t}^h &= (1 - s_2^y)y_{t-1}^h + (1 - s_2^v)w_{t-1}^h
\end{aligned}
$$

- Assume that $s_1^y \leq s_2^y$ and $s_1^v \leq s_2^v$, which implies that workers save less than investors.

- Household's saving is thus the difference between disposable income and consumption:

$$
\begin{aligned}
s_{1,t}^h &= y_{1,t}^h - c_{1,t}^h \\
s_{2,t}^h &= y_{2,t}^h - c_{2,t}^h
\end{aligned}
$$

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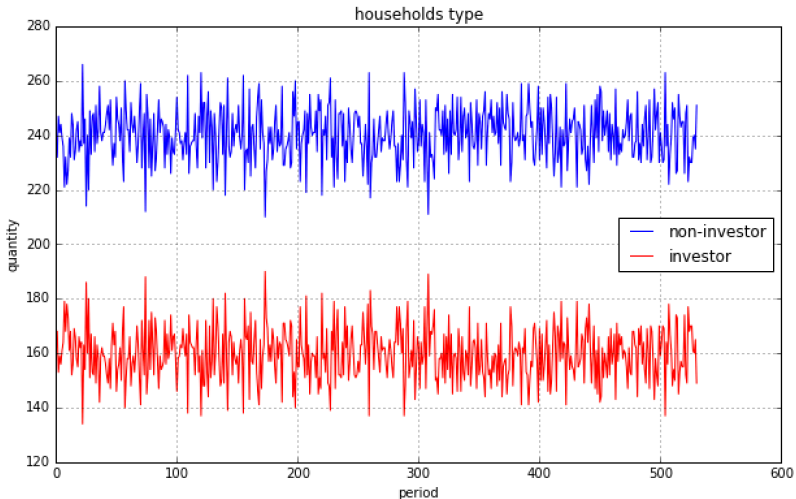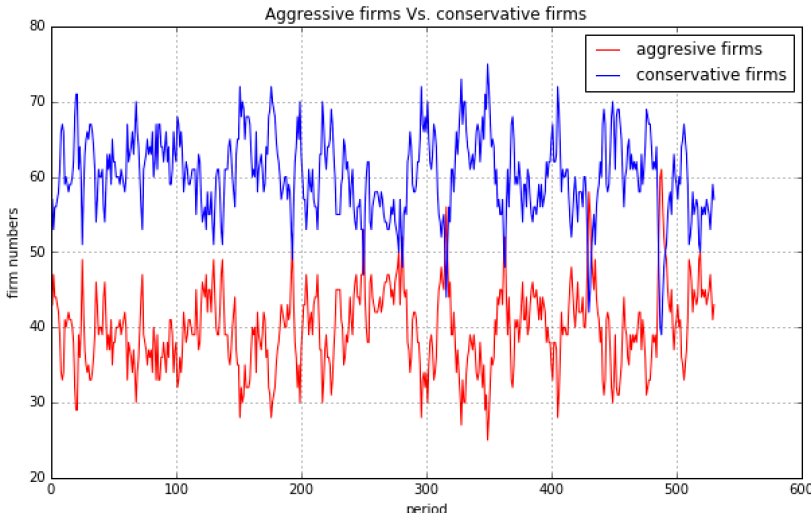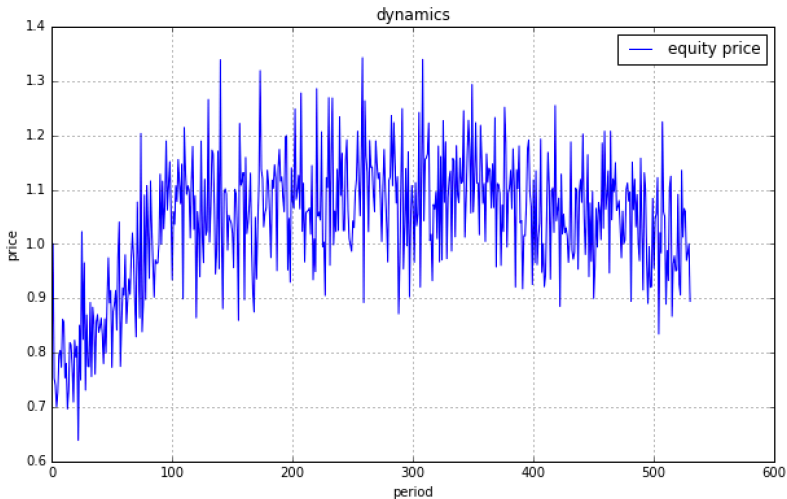
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Equity market

- Assume that firm n raises external funds according to the proportions

$$b_{t+1}^n - b_t^n = \varpi(i_{t+1}^n - a_{t+1}^n)$$
$$p_{t+1}^e(e_{t+1}^n - e_t^n) = (1 - \varpi)(i_{t+1}^n - a_{t+1}^n),$$

where $0 \le \varpi \le 1$ is a constant common to all firms.

- Conversely, assume that the demand for equities for household m is given by

$$p_{t+1}^e e_{t+1}^m = \varphi v_{t+1}^m(z_{t+1}^m - 1) = \begin{cases} 0 & \text{if } z_{t+1}^m = 1 \\ \varphi v_{t+1}^m & \text{if } z_{t+1}^m = 2, \end{cases}$$

where φ is a constant common to all households.

Aggressive firms Vs. conservative firms

households type

households type

Aggressive firms Vs. conservative firms

Speculative, ponzi, hedge firms

Aggressive firms Vs. conservative firms

Example 2: household income

Speculative, ponzi, hedge firms

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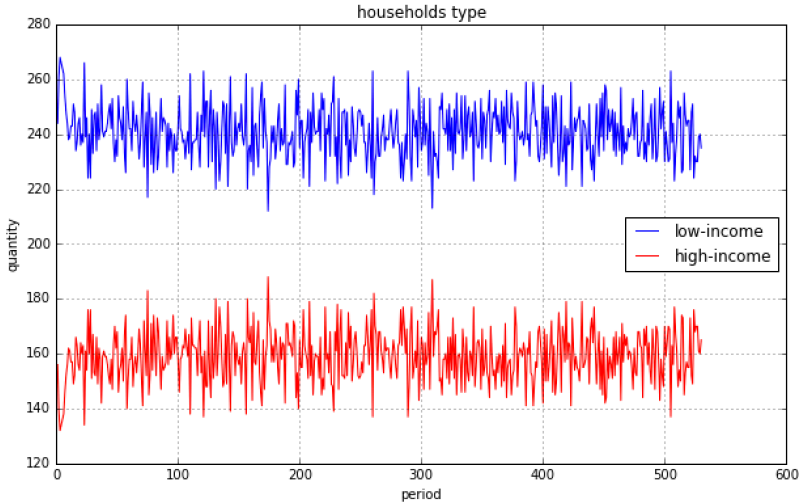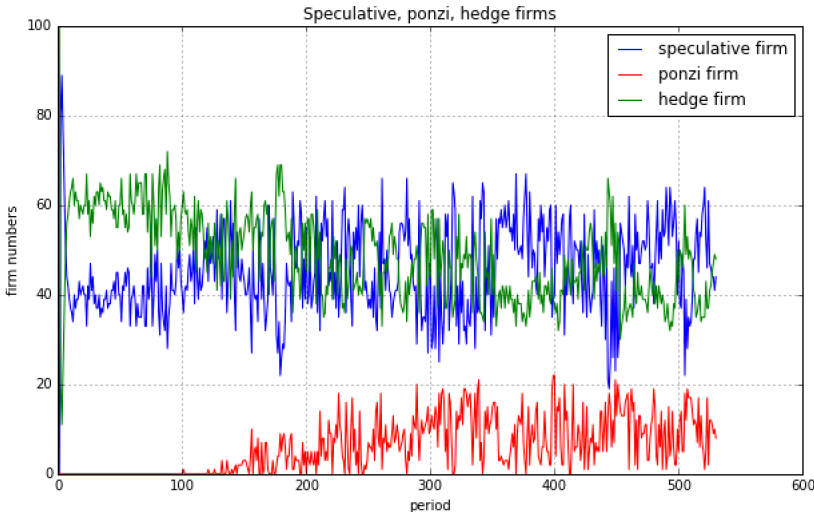approaches

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Concluding remarks

- Macroeconomics is too important to be left to macroeconomists.
- Since Keynes's death it has developed in two radically different approaches:
  1. The dominant one has the appearance of mathematical rigour (the SMD theorems notwithstanding), but is based on implausible assumptions, has poor fit to data in general, and is disastrously wrong during crises. Finance plays a negligible role
  2. The heterodox approach is grounded in history and institutional understanding, takes empirical work much more seriously, but is generally averse to mathematics. Finance plays a major role.
- It's clear which approach should be embraced by mathematical finance.