Nonlinearity, correlation and the valuation of employee stock options

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1. Introduction

We consider an employee who has been awarded a compensation package consisting of A identical call options on the company's stock with the following features:

- strike price K, maturity date T and vesting period $T_v < T$;
- options are non-transferible;
- hedge using the underlying stock Y_t is not allowed;
- hedge using a correlated asset S_t is allowed.

2. Accounting reccomendations

- The *Financial Accounting Standard Board* instructed in 1972 (Opinion 25) that stock options should be accounted according to their intrinsic value, that is $(Y_t - K)^+$ on the date their are granted.

- In 1995, the FASB 123 recommended using a fair value approach instead: estimate the expected life of the option and insert this into either Black–Scholes or a Cox–Rubenstein-Ross tree. It still accepted Opinion 25 as a valid method.

- In 2004, it revised FASB 123, eliminating the possibility of using intrinsic value methods.

3. Previous literature

- Detemple and Sudaresan (1999) and Hall and Murphy (2002) propose to use utility methods to deal with the market incompleteness created by trading and hedging restrictions, but without using a correlated asset.

- Musiela and Zariphopoulou (2004) developed a multiperiod model to price European style contracts based on a non-traded underlying asset in the presence of a correlated traded asset using indifference pricing techniques.

- Henderson (2005) applied indifference pricing to value a single American call options on a non-traded asset.

- Rogers and Scheinkman (2003) and Jain and Subramanian (2004) investigate the effect of partial exercise, but with no correlated asset.

- Hull and White (2004) use a binomial model with no correlated asset, no partial exercise and no risk preferences. The incompleteness is accounted for by a parameter M - the *effective stock-to-strike* exercise threshold.

4. The one-period model

Consider a one-period market model

$$(S_T, Y_T) = \begin{cases} (uS_0, hY_0) & \text{with probability } p_1, \\ (uS_0, \ell Y_0) & \text{with probability } p_2, \\ (dS_0, hY_0) & \text{with probability } p_3, \\ (dS_0, \ell Y_0) & \text{with probability } p_4, \end{cases} \tag{1}$$

where $0 < d < 1 < u$ and $0 < \ell < 1 < h$, for positive initial values S_0, Y_0 and historical probabilities p_1, p_2, p_3, p_4

Let $C_T = C(Y_T)$ be a T–claim and consider a utility function $U(x) = -e^{-\gamma x}$. An investor who *buys* this claim for a price π will then try to solve the optimal portfolio problem

$$u^C(x - \pi) = \sup_H E[U(X_T + C_T)]. \qquad (2)$$

The indifference price for this claim is defined to be a solution to the equation

$$u^0(x) = u^C(x - \pi),$$

where u^0 is defined by (2) for the degenerate case $C \equiv 0$.

An explicit calculation then leads to

$$\pi = g(C_h, C_\ell) \tag{3}$$

where, for fixed parameters $(u, d, p_1, p_2, p_3, p_4)$ the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$g(x_1, x_2) = \frac{q}{\gamma} \log \left(\frac{p_1 + p_2}{p_1 e^{-\gamma x_1} + p_2 e^{-\gamma x_2}} \right) + \frac{1-q}{\gamma} \log \left(\frac{p_3 + p_4}{p_3 e^{-\gamma x_1} + p_4 e^{-\gamma x_2}} \right),$$

with

$$q = \frac{1-d}{u-d}.$$

Now suppose C is an American claim. It is clear that early exercise will occur whenever

$$C(Y_0) \geq \pi,$$

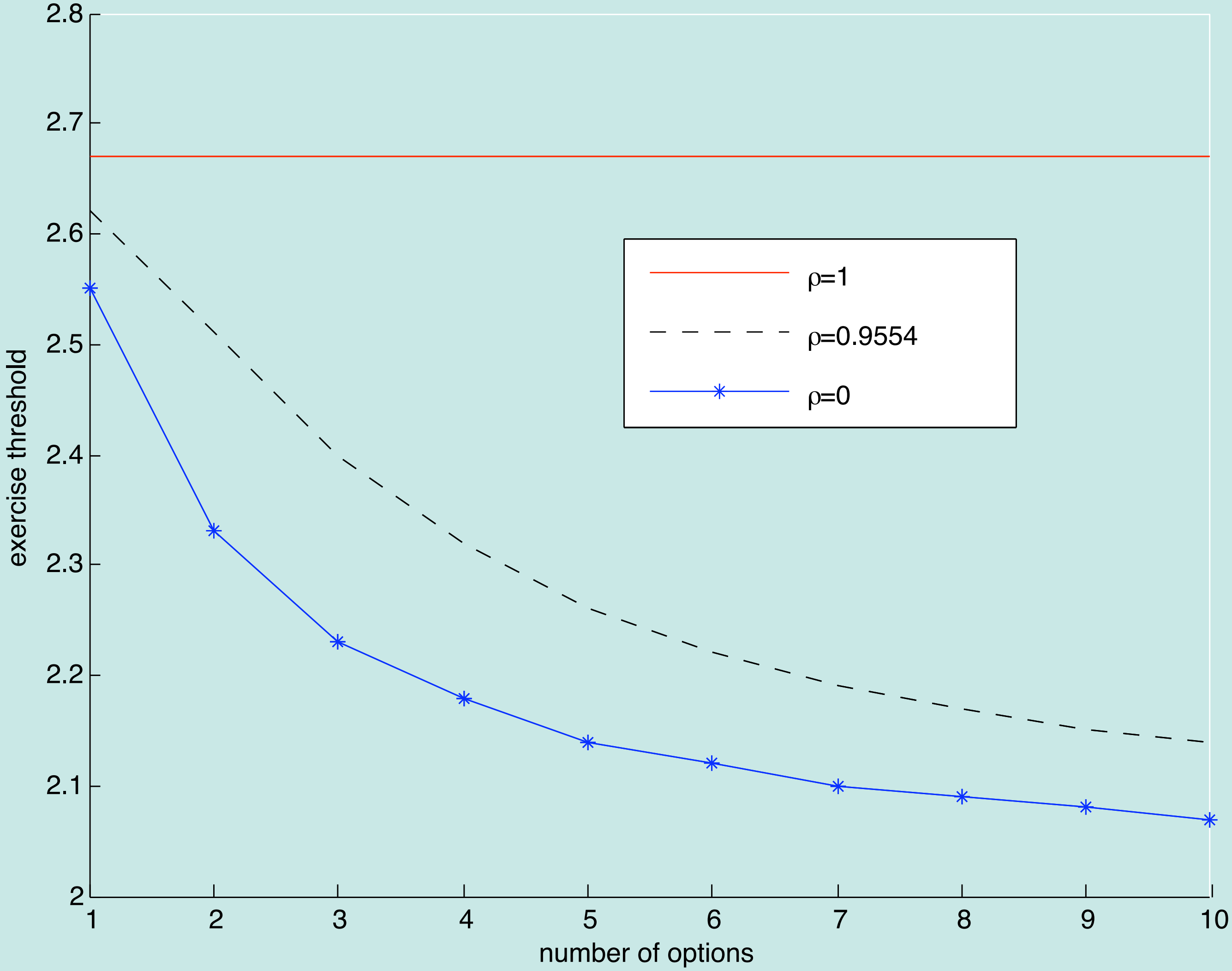
where π^B is the (European) indifference price. For example, an American call option with strike price K will be exercised if Y_0 exceeds the solution to

$$(Y^* - K)^+ = g((hY_0 - K)^+, (\ell Y_0 - K)^+)$$

5. Multiple claims

As a result of risk aversion, the early exercise threshold for an American call option obtained above is different (and higher) than the exercise threshold for a contract consisting of A units of identical Americal calls. Explicitly, it is the solution to

$$A(Y^* - K)^+ = g(A(hY_0 - K)^+, A(\ell Y_0 - K)^+) \qquad (4)$$

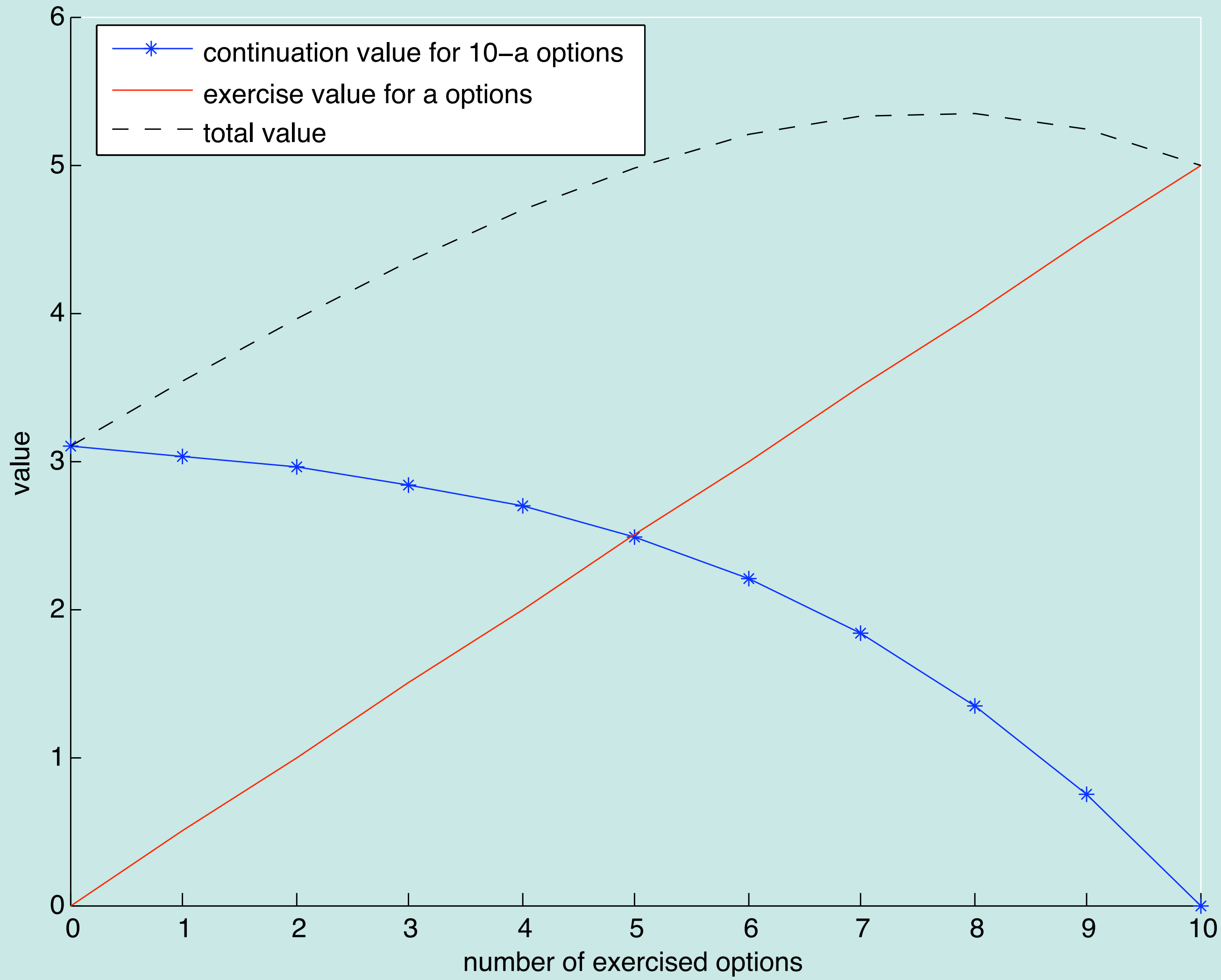

2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2
exercise threshold
1
2
3
4
5
6
7
8
9
10
number of options
ρ=1
ρ=0.9554
ρ=0

If partial exercise is allowed, then the optimal number of options to be exercised is the solution a^* to

$$\max_a \left[a(Y_0 - K)^+ + \pi^{(A-a)B} \right]. \tag{5}$$

The value of A units of the option is therefore

$$C_0^{(A)} = a_0(Y_0 - K)^+ + \pi^{(A-a_0)}$$


continuation value for 10−a options
exercise value for a options
total value
value
number of exercised options
0
1
2
3
4
5
6
7
8
9
10

6. Two-period model: inter-temporal exercise

Let us label the nodes in of a tow-period binomial tree by 0 at time $t_0 = 0$, (h, ℓ) at time t_1 and $(hh, h\ell, \ell\ell)$ at time $t_2 = T$.

The number of option that the holder of A calls at the node h should immediately exercise is given by

$$a_h = \arg\max_{0 \le a \le A} \left[a(hY_0 - K)^+ + \pi_h^{(A-a)} \right], \tag{6}$$

where $\pi_h^{(A-a)}$ denotes the indifference of an European claim to starting at the node h and maturing at time T.

The pay-offs for such claim are with

$$C_{hh}^{(A-a)} = (A-a)(hhY_0 - K)^+$$

with probability $(p_1 + p_3)$ and

$$C_{h\ell}^{(A-a)} = (A-a)(h\ell Y_0 - K)^+$$

with probability $(p_2 + p_4)$, where we have used hh and $h\ell$ to denoted, respectively, the nodes where the non-traded asset has values hhY_0 and $h\ell Y_0$. Its indifference price is explicitly given by

$$\pi_h^{(A-a)} = g(C_{hh}^{(A-a)}, C_{h\ell}^{(A-a)}) \tag{7}$$

In the same vein, the optimal number of options to be exercised at the node ℓ, where the non-trade asset has value ℓY_0, is

$$a_\ell = \arg\max_{0 \le a \le A} \left[a(\ell Y_0 - K)^+ + \pi_\ell^{(A-a)} \right], \tag{8}$$

where

$$\pi_\ell^{(A-a)} = g(C_{h\ell}^{(A-a)}, C_{\ell\ell}^{(A-a)}) \tag{9}$$

Therefore, at the intermediate time t_1, the total value of A options at the node h is

$$C_h^{(A)} := \left[a_h(hY_0 - K)^+ + \pi_{h1}^{(A-a_h)} \right], \tag{10}$$

while the total value of A options at the node ℓ is

$$C_\ell^{(A)} := \frac{1}{A} \left[a_\ell(\ell Y_0 - K)^+ + \pi_{\ell 1}^{(A-a_\ell)} \right]. \tag{11}$$

Finally, starting with A units of the option, the number of options that should be exercised at the initial time t_0 is

$$a_0 = \arg\max_{0 \le a \le A} \left[a(Y_0 - K)^+ + \pi_0^{(A-a)} \right], \tag{12}$$

where

$$\pi_h^{(A-a)} = g(C_h^{(A-a)}, C_\ell^{(A-a)}) \tag{13}$$

Therefore the value at time zero of A units of an American call option on the non-traded asset is

$$C_0^{(A)} := \left[a_0(Y_0 - K)^+ + \pi_0^{(A-a_0)} \right]. \tag{14}$$

6. The multi-period model: inter-temporal exercise

We first have to choose discrete time parameters $(u, d, h, \ell, p_1, p_2, p_3, p_4)$ that match the distributional properties of the continuos time diffusion

$$\begin{aligned} dS &= (\mu - r)Sdt + \sigma S dW && (15) \\ dY &= (a - r - \delta)Y dt + bY(\rho dW + \sqrt{1-\rho^2})dZ, && (16) \end{aligned}$$

These are given by the system

$$
\begin{aligned}
u &= e^{\sigma\sqrt{\Delta t}}, \qquad h = e^{b\sqrt{\Delta t}} \\
d &= e^{-\sigma\sqrt{\Delta t}}, \qquad \ell = e^{-b\sqrt{\Delta t}} \\
p_1 + p_2 &= \frac{e^{(\mu - r)\Delta t} - d}{u - d} \\
p_1 + p_3 &= \frac{e^{(a - r - \delta)\Delta t} - \ell}{h - \ell} \\
\rho b \sigma \Delta t &= (u - d)(h - \ell)[p_1 p_4 - p_2 p_3] \\
1 &= p_1 + p_2 + p_3 + p_4
\end{aligned}
$$

The valuation algorithm is then:

- Begin at the final period.
- At each node of the tree, compute the (European) indifference prices for different values of $(A - a)$.
- Determining the maximum of (5).
- Use this as the value for the entire position at that node.
- Iterate backwards.

7. Numerical Results

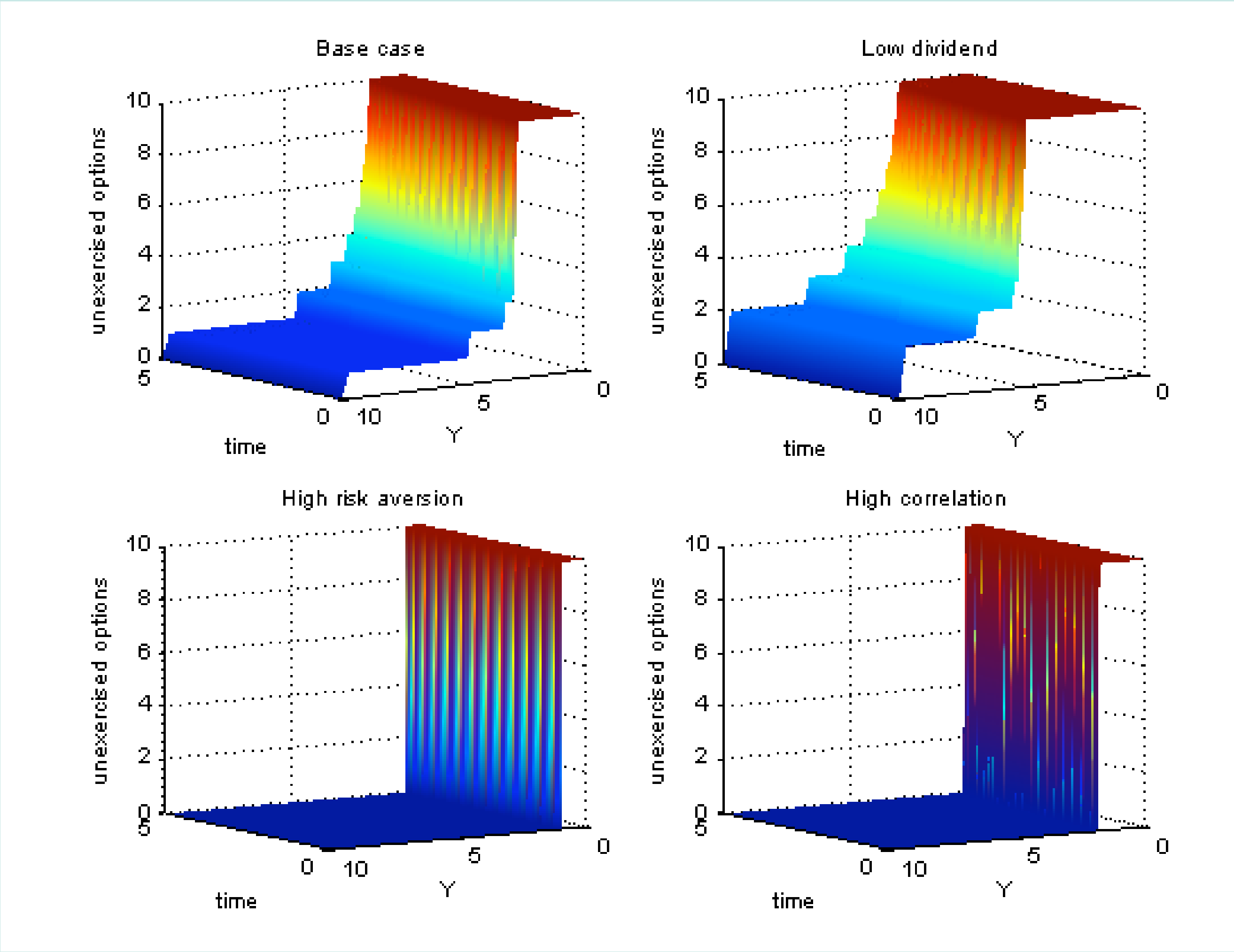
We first determine the optimal exercise surface for the holder of $A = 10$ options with strike price $K = 1$ and

$$\mu = 0.12, \quad \sigma = 0.2, \quad S_0 = 1 \tag{17}$$

$$a = 0.15 \quad b = 0.3, \quad Y_0 = 1 \tag{18}$$

$$r = 0.06 \quad T = 5, \quad N = 500 \tag{19}$$

For our base case, $\delta = 0.075$, $\gamma = 0.125$ and $\rho = -0.5$. We then modify it by having $\delta = 0$, $\gamma = 10$ and $\rho = 0.95$.



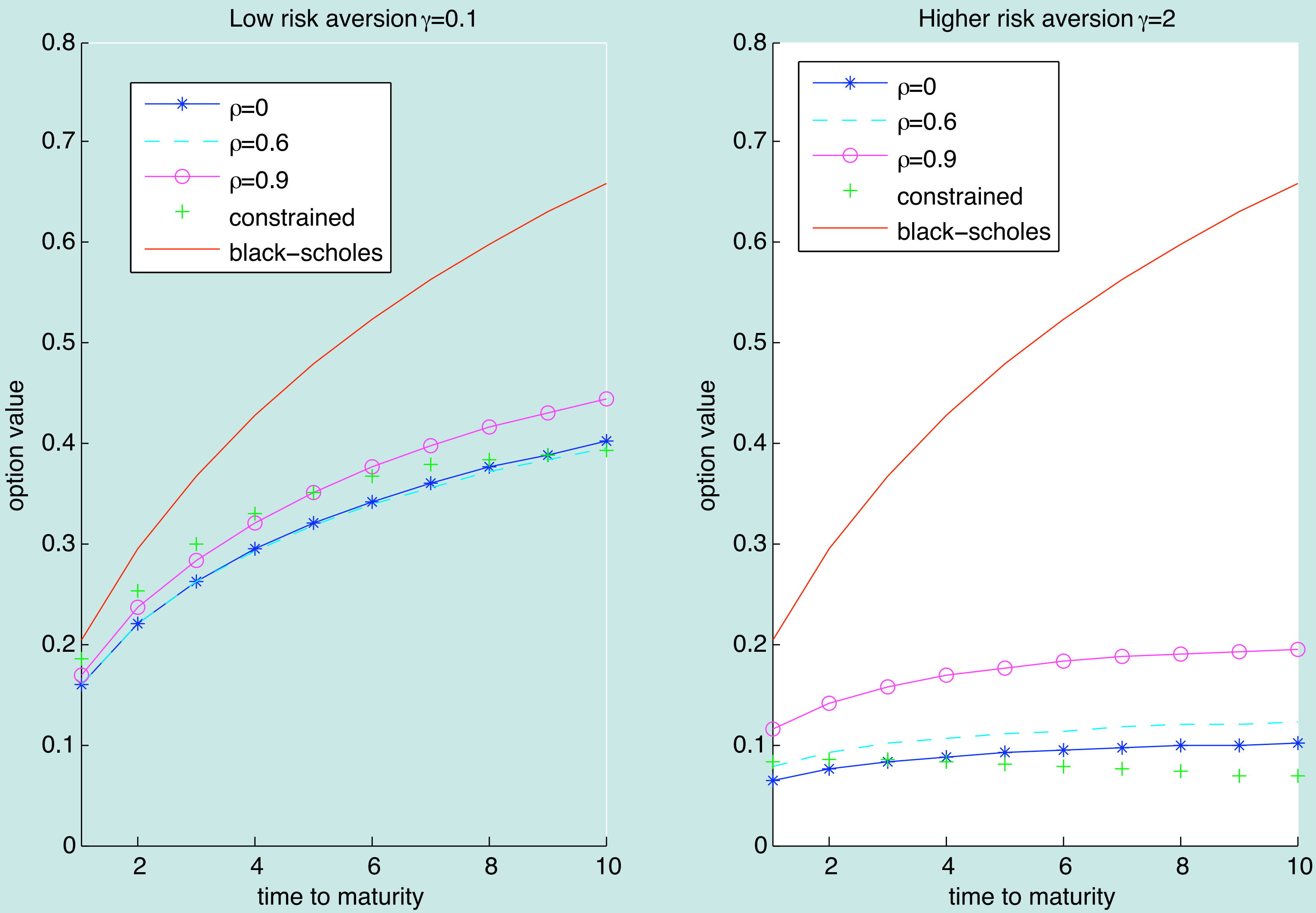
Base case
Low dividend
High risk aversion
High correlation
unexercised options
time
Y

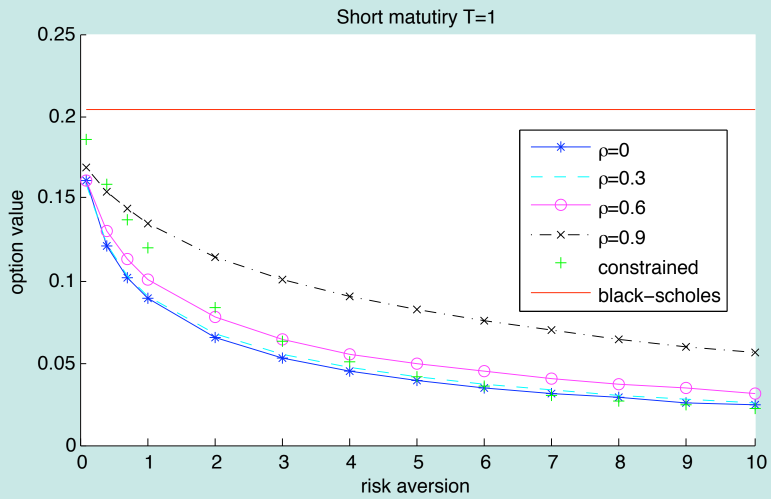
Next we consider the impact that time-to-maturity, risk aversion, correlation and volatility have on the option price. When not indicated in the graphs, the parameter values are

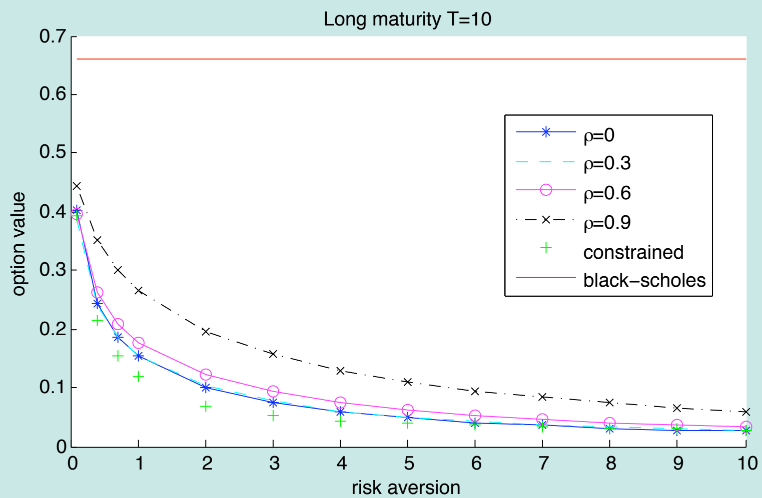
$$\mu = 0.09, \quad \sigma = 0.4, \quad S_0 = 1 \tag{20}$$

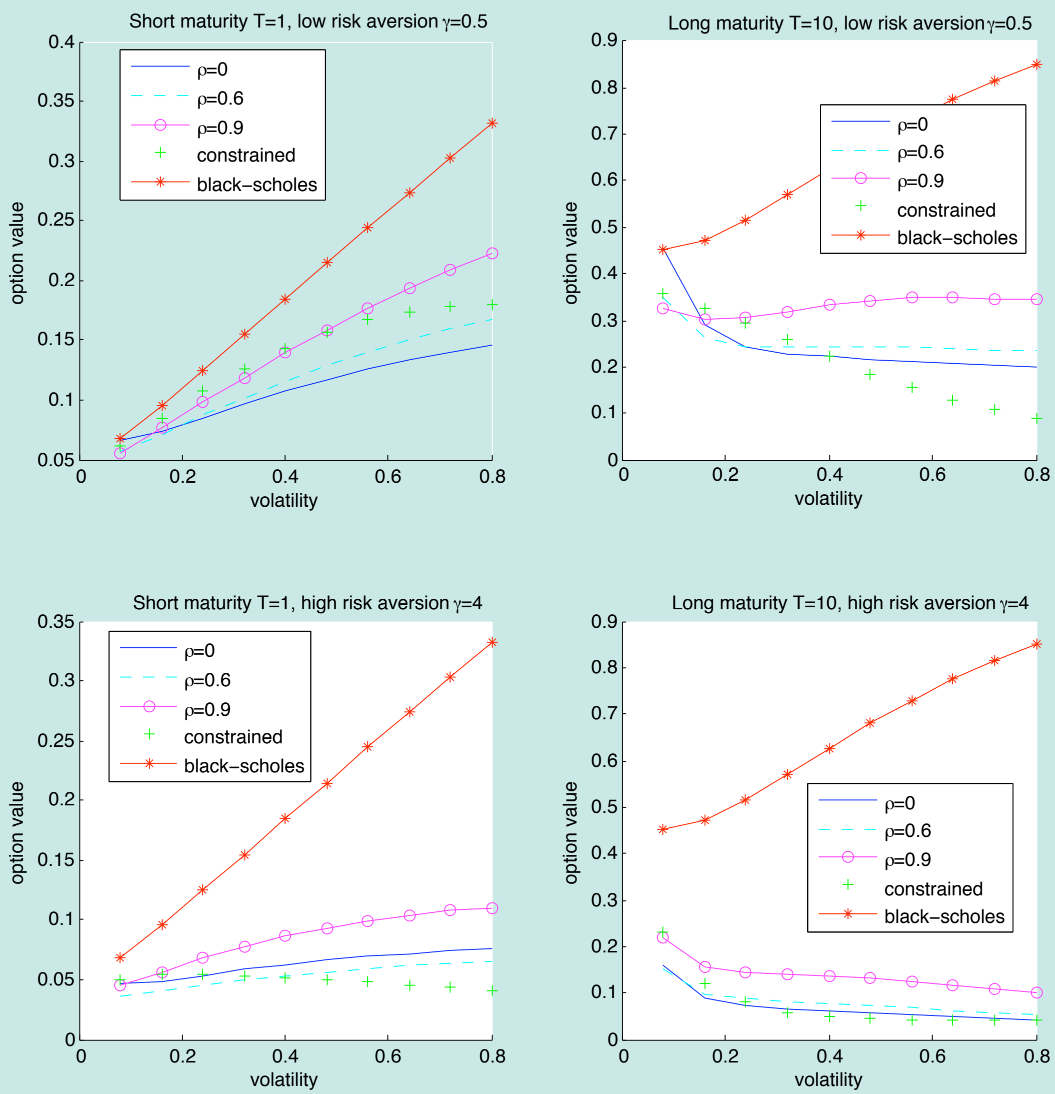
$$a = 0.08 \quad b = 0.45, \quad Y_0 = 1 \tag{21}$$

$$r = 0.06 \quad \delta = 0, \quad N = 100 \tag{22}$$

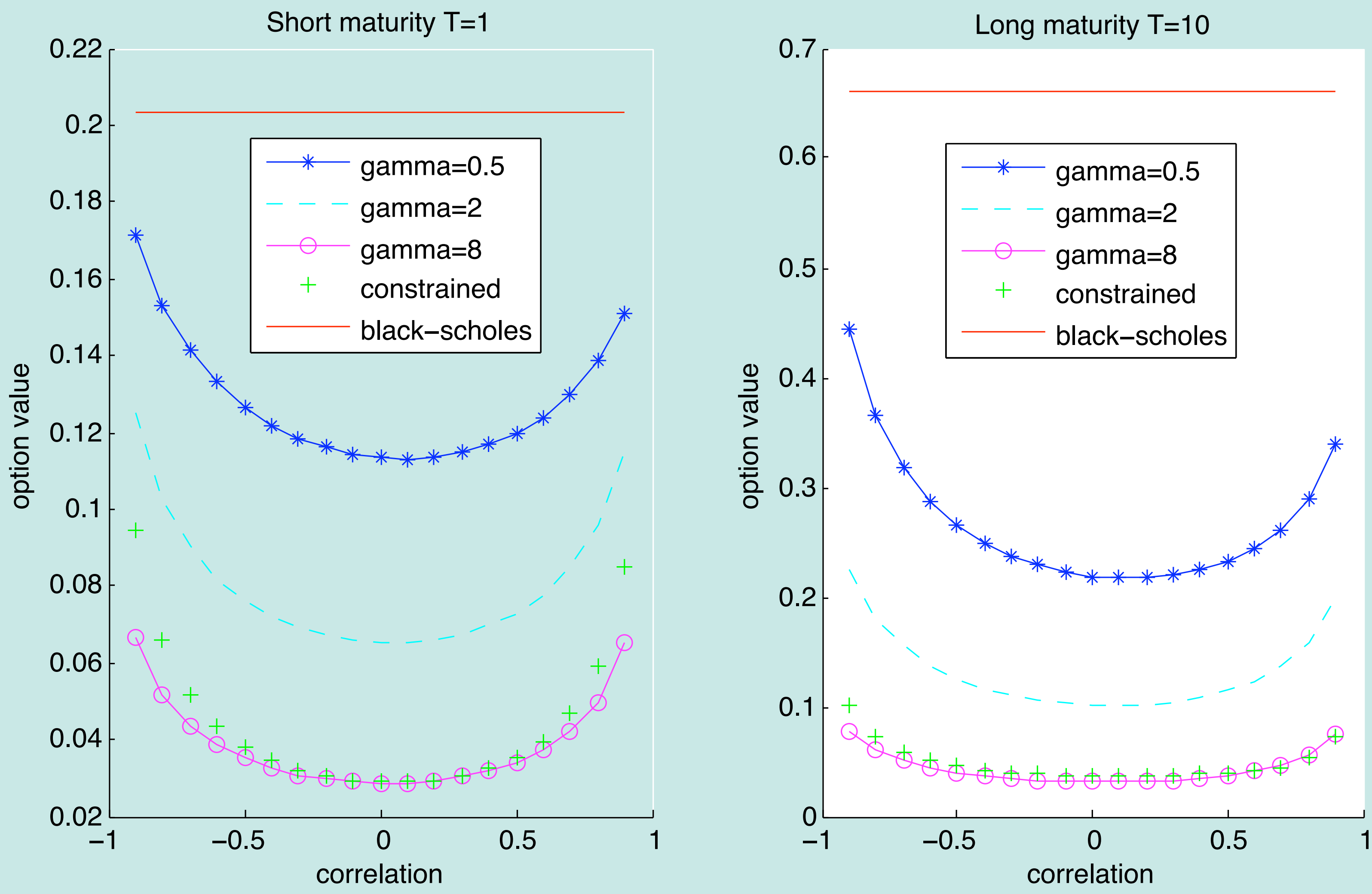

Low risk aversion γ=0.1
Higher risk aversion γ=2
option value
time to maturity
ρ=0
ρ=0.6
ρ=0.9
constrained
black−scholes
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
2
4
6
8
10

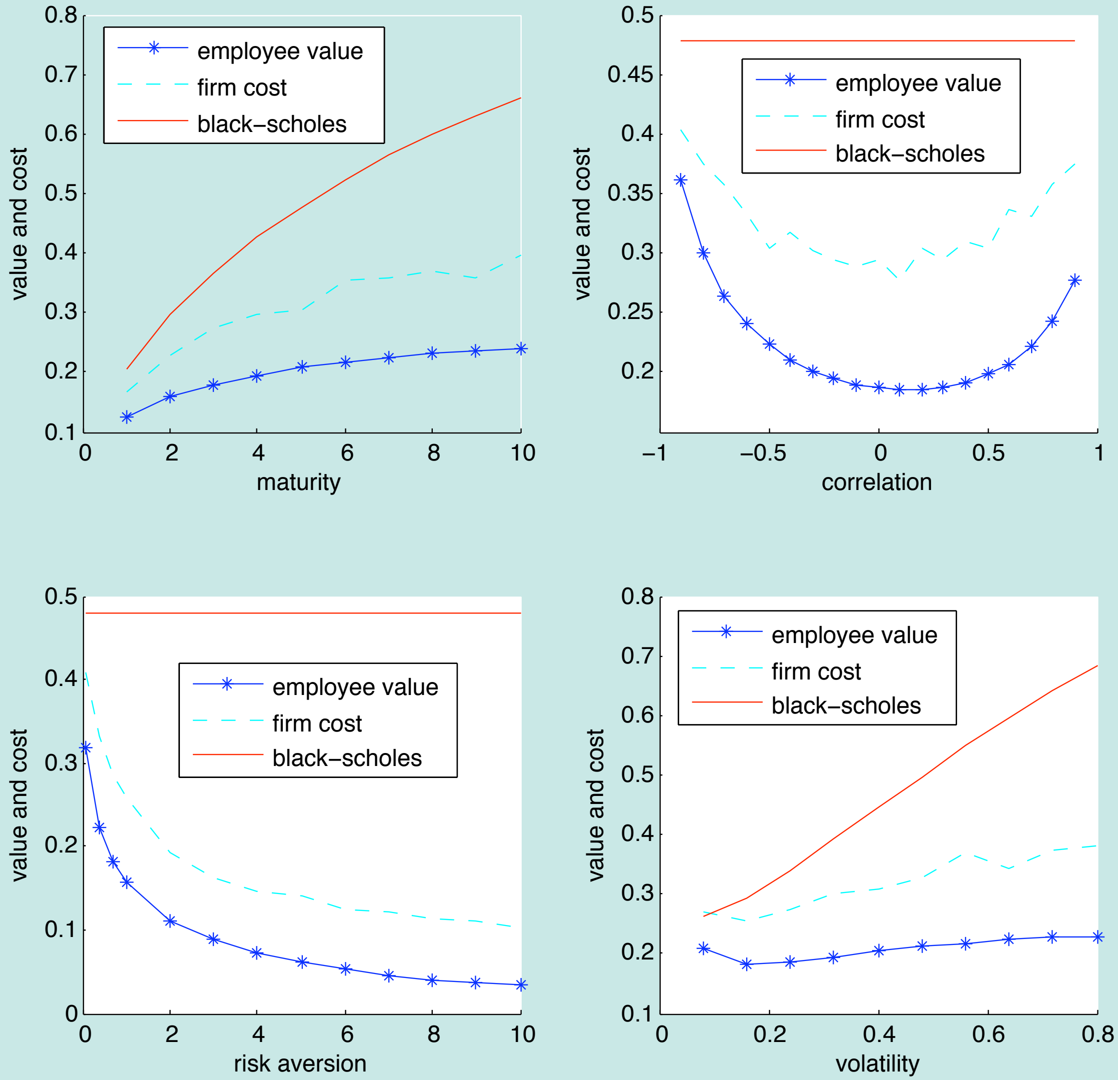

Short matutiry T=1
option value
risk aversion
ρ=0
ρ=0.3
ρ=0.6
ρ=0.9
constrained
black−scholes


Long maturity T=10
option value
risk aversion
ρ=0
ρ=0.3
ρ=0.6
ρ=0.9
constrained
black−scholes



Short maturity T=1, low risk aversion γ=0.5
Long maturity T=10, low risk aversion γ=0.5
Short maturity T=1, high risk aversion γ=4
Long maturity T=10, high risk aversion γ=4
option value
volatility
ρ=0
ρ=0.6
ρ=0.9
constrained
black–scholes


Short maturity T=1
option value
correlation
gamma=0.5
gamma=2
gamma=8
constrained
black–scholes
Long maturity T=10
option value
correlation
gamma=0.5
gamma=2
gamma=8
constrained
black–scholes


employee value
firm cost
black–scholes
value and cost
maturity
correlation
risk aversion
volatility