The investment game in incomplete markets

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- temporarily suspend operations under adverse conditions.

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- get married;
- change the constitution of a country;
- introduce environmental laws;
- develop a controversial highway;
- commit suicide !

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- In general, competition erodes the value of flexibility.

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- Finally, competition is generally introduced using game theory.
- Surprisingly, game theory is almost exclusively combined with real options under the hypothesis of risk-neutrality !

A one–period investment model

- Consider a two–factor market where the discounted prices for the project V and a correlated traded asset S follow:

$$(S_T, V_T) = \begin{cases} (uS_0, hV_0) & \text{with probability } p_1, \\ (uS_0, \ell V_0) & \text{with probability } p_2, \\ (dS_0, hV_0) & \text{with probability } p_3, \\ (dS_0, \ell V_0) & \text{with probability } p_4, \end{cases} \qquad (1)$$

where $0 < d < 1 < u$ and $0 < \ell < 1 < h$, for positive initial values S_0, V_0 and historical probabilities p_1, p_2, p_3, p_4.

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- Let the risk preferences be specified through an exponential utility $U(x) = -e^{-\gamma x}$.
- An investment opportunity is model as an option with discounted payoff $C_t = (V - e^{-rt} I)^+$, for $t = 0, T$.

European Indifference Price

- The indifference price for the option to invest in the final period as the amount π that solves the equation

$$\max_H E[U(x + H(S_T - S_0)] = \max_H E[U(x - \pi + H(S_T - S_0) + C_T]$$

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- Denoting the two possible pay-offs at the terminal time by C_h and C_ℓ, the European indifference price is explicitly given by

$$\pi = g(C_h, C_\ell) \qquad (2)$$

where, for fixed parameters $(u, d, p_1, p_2, p_3, p_4)$ the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined as

$$\begin{aligned} g(x_1, x_2) &= \frac{q}{\gamma} \log\left(\frac{p_1 + p_2}{p_1 e^{-\gamma x_1} + p_2 e^{-\gamma x_2}}\right) \\ &\quad + \frac{1-q}{\gamma} \log\left(\frac{p_3 + p_4}{p_3 e^{-\gamma x_1} + p_4 e^{-\gamma x_2}}\right), \end{aligned} \qquad (3)$$

with

$$q = \frac{1-d}{u-d}.$$

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- That is, from the point of view of this agent, the value at time zero for the opportunity to invest in the project either at $t = 0$ or $t = T$ is given by

$$C_0 = \max\{(V_0 - I)^+, g((hV_0 - e^{-rT} I)^+, (\ell V_0 - e^{-rT} I)^+)\}.$$

A multi–period model

- Consider now a continuous-time two–factor market of the form

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\begin{aligned}
dS_t &= (\mu_1 - r)S_t dt + \sigma_1 S_t dW \\
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- We want to approximate this market by a discrete–time processes (S_n, V_n) following the one–period dynamics (1).
- This leads to the following choice of parameters:

$$\begin{aligned} u &= e^{\sigma_1\sqrt{\Delta t}}, \qquad h = e^{\sigma_2\sqrt{\Delta t}}, \\ d &= e^{-\sigma_1\sqrt{\Delta t}}, \qquad \ell = e^{-\sigma_2\sqrt{\Delta t}}, \\ p_1 + p_2 &= \frac{e^{(\mu_1-r)\Delta t} - d}{u-d}, \quad p_1 + p_3 = \frac{e^{(\mu_2-r)\Delta t} - \ell}{h-\ell} \\ \rho\sigma_1\sigma_2\Delta t &= (u-d)(h-\ell)[p_1p_4 - p_2p_3], \end{aligned}$$

supplemented by the condition $p_1 + p_2 + p_3 + p_4 = 1$.

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- Given these parameters, the CAPM equilibrium expected rate of return on the project for a given correlation ρ is

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- The difference $\delta = \bar{\mu}_2 - \mu_2$ is the below–equilibrium rate–of–return shortfall and plays the role of a dividend rate paid by the project, which we fix at $\delta = 0.04$.

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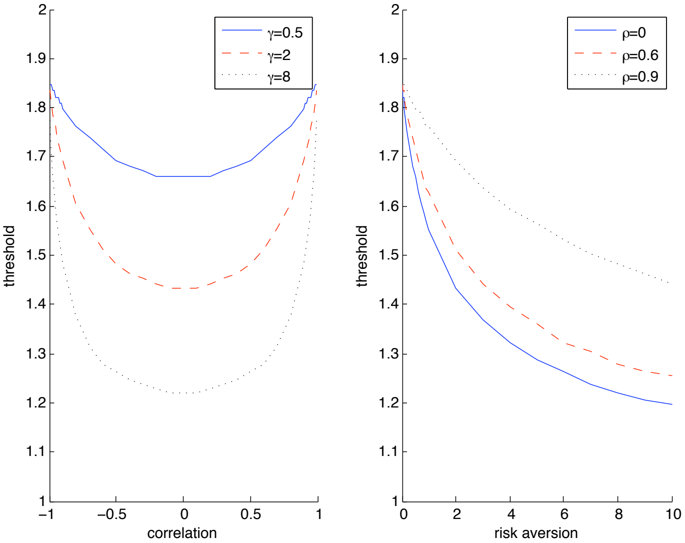
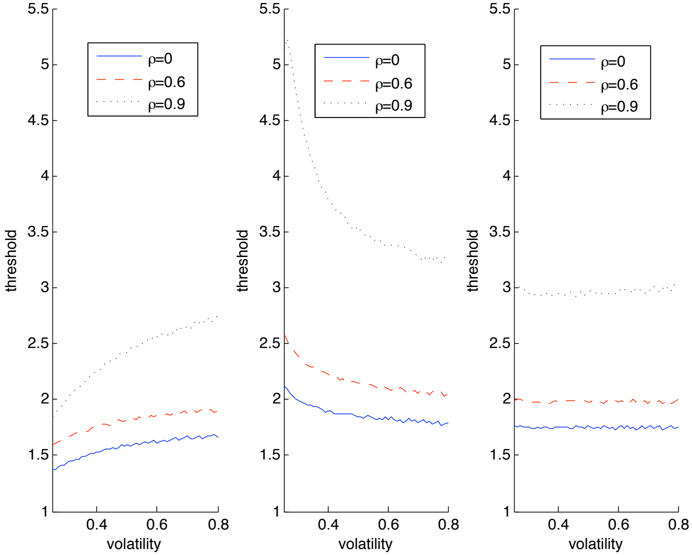
- For our parameters, the adjustment to market risks is accounted by CAPM and this threshold coincides with $V^*_{DP} = 2$

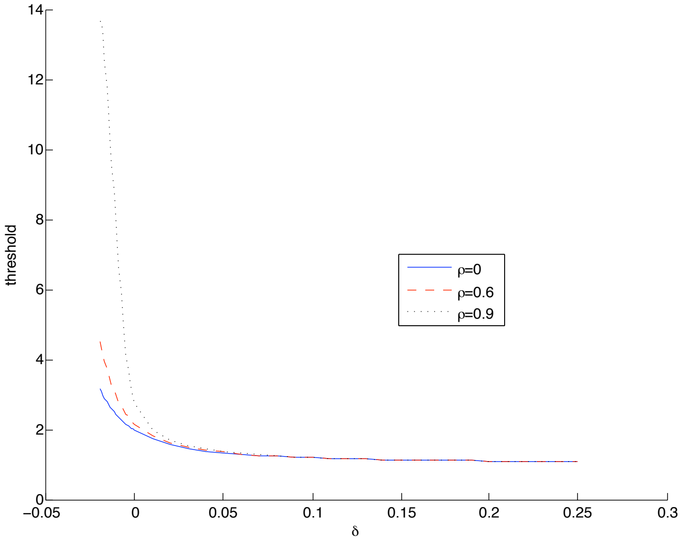
Dependence with Correlation and Risk Aversion



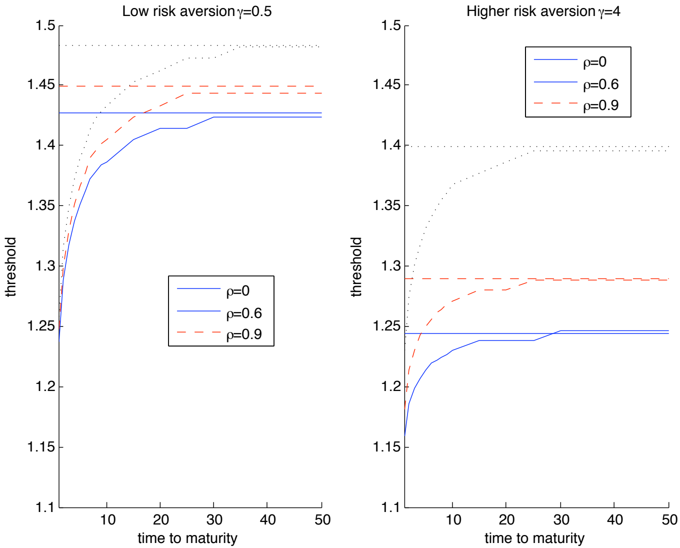
Dependence with Volatility



Dependence with Dividend Rate



Dependence with Time to Maturity



Values for the option to invest

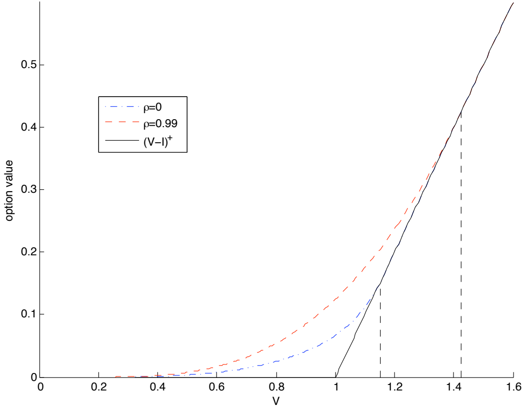


Figure: Option value as a function of underlying project value. The threshold for $\rho = 0$ is 1.1972 and the one for $\rho = 0.99$ is 1.7507.

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  2. Once the solution for a given game is found on a decision node, its value becomes the pay-off for an option at that node.
- In this way, option valuation and game theoretical equilibrium become dynamically related in a decision tree.

One–period expansion option under monopoly

- Suppose now that a firm faces the decision to expand capacity for a product with uncertain demand:

$$Y_1 = \begin{cases} hY_0 & \text{with probability } p \\ \ell Y_0 & \text{with probability } 1-p \end{cases}, \tag{5}$$

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- The cash flow per unit demand for the firm is denoted by $D_{x(k)}$.

The NPV solution

- If no expansion occurs at time t_0, then the value of the project at time t_0 is

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- textcolorredIf the decision needs to be taken at time t_0, then according to NPV the firm should expand provided $v_{in} \geq v_{out}$, that is, if the sunk cost I is smaller then

$$I^{NPV} = (D_1 - D_0) Y_0 + (\pi_0(D_1 Y_1) - \pi_0(D_0 Y_1)). \quad (7)$$

The RO solution

- By contrast, if the decision to invest can be postponed until time t_1, then the value of the project when no investment occurs at time t_0 is

$$v_{wait} = D_0 Y_0 + \pi_0(C_1),$$

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- Accordingly, the firm should invest at time t_0 provided $v_{in} \geq v_{wait}$, that is, if the sunk cost is smaller than

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- Since the function g is non-decreasing in each of its arguments,

$$I^{NPV} - I^{RO} = \pi_0(C_1) - \pi_0(D_0 Y_1) \geq 0. \tag{9}$$

The RO solution

- By contrast, if the decision to invest can be postponed until time t_1, then the value of the project when no investment occurs at time t_0 is

$$v_{wait} = D_0 Y_0 + \pi_0(C_1),$$

where C_1 denotes the random variable

$$C_1 = C_1(Y_1) = \max\{D_0 Y_1, D_1 Y_1 - I\} \geq D_0 Y_1.$$

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- That is, according to RO, the firm is less likely to expand at time t_0.

One–period duopoly

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Equilibrium strategies

Lemma

Under FMA:

1. *If* $I < I_F^h$*, then the equilibrium strategy is* $(1, 1)$ *for high and low demand.*
2. *If* $I_F^\ell < I < I_F^h$ *and* $I < I_L^\ell$*, then the equilibrium strategy is* $(1, 1)$ *for high demand and* $(1, 0)$ *for low demand.*
3. *If* $I_F^h < I < I_L^\ell$*, then the equilibrium strategy is* $(1, 0)$ *for high and low demand.*
4. *If* $I_F^\ell < I < I_F^h$ *and* $I_L^\ell < I$*, then the equilibrium strategy is* $(1, 1)$ *for high demand and* $(0, 0)$ *for low demand.*
5. *If* $I_L^\ell < I < I_L^h$ *and* $I_F^h < I$*, then the equilibrium strategy is* $(1, 0)$ *for high demand and* $(0, 0)$ *for low demand.*
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A multi-period investment game

- Consider two firms L and F each operating a project with an option to re-invest at cost I and increase cash–flow according to an uncertain demand

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW.$$

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Derivation of project values (1)

- Let $V_i^{(x_i(t_{m-1}),x_j(t_{m-1}))}(t_m, y)$ denote the project value for firm i at time t_m and demand level y.

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$$
\begin{aligned}
v_i^{(1,1)}(t_m, y) &= D_{11} y \Delta t + \frac{g(V_i^{(1,1)}(t_{m+1}, y^u), (V_i^{(1,1)}(t_{m+1}, y^d))}{e^{r\Delta t}} \\
v_L^{(1,0)}(t_m, y) &= D_{10} y \Delta t + \frac{g(V_L^{(1,0)}(t_{m+1}, y^u), (V_L^{(1,0)}(t_{m+1}, y^d))}{e^{r\Delta t}} \\
v_L^{(0,1)}(t_m, y) &= D_{01} y \Delta t + \frac{g(V_L^{(0,1)}(t_{m+1}, y^u), (V_L^{(0,1)}(t_{m+1}, y^d))}{e^{r\Delta t}} \\
v_F^{(1,0)}(t_m, y) &= D_{01} y \Delta t + \frac{g(V_F^{(1,0)}(t_{m+1}, y^u), (V_F^{(1,0)}(t_{m+1}, y^d))}{e^{r\Delta t}} \\
v_F^{(0,1)}(t_m, y) &= D_{10} y \Delta t + \frac{g(V_F^{(0,1)}(t_{m+1}, y^u), (V_F^{(0,1)}(t_{m+1}, y^d))}{e^{r\Delta t}} \\
v_i^{(0,0)}(t_m, y) &= D_{00} y \Delta t + \frac{g(V_i^{(0,0)}(t_{m+1}, y^u), (V_i^{(0,0)}(t_{m+1}, y^d))}{e^{r\Delta t}}
\end{aligned}
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- For fully invested firms, the project values are simply given by

$$V_i^{(1,1)}(t_m, y) = v_i^{(1,1)}(t_m, y).$$

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$$V_L^{(0,1)}(t_m, y) = \max\{v_L^{(1,1)}(t_m, y) - I, v_L^{(0,1)}(t_m, y)\}.$$

Derivation of project values (3)

- Next consider the project value for L when it has already invest and F hasn't:

$$V_L^{(1,0)}(t_m, y) = \begin{cases} v_L^{(1,1)}(t_m, y) \text{ if } v_F^{(1,1)}(t_m, y) - I > v_F^{(1,0)}(t_m, y), \\ v_L^{(1,0)}(t_m, y) \text{ otherwise.} \end{cases}$$

Derivation of project values (3)

- Next consider the project value for L when it has already invest and F hasn't:

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- Similarly, the project value for F when it has already invest and L hasn't is

$$V_F^{(0,1)}(t_m, y) = \begin{cases} v_F^{(1,1)}(t_m, y) \text{ if } v_L^{(1,1)}(t_m, y) - I > v_L^{(0,1)}(t_m, y), \\ v_F^{(0,0)}(t_m, y) \text{ otherwise.} \end{cases}$$

Derivation of project values (4)

- Finally, the project values $V_i^{(0,0)}$ are obtained as a Nash equilibrium, since both firms still have the option to invest.

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- The pay-off matrix for the game is

		Firm F	
		Invest	Wait
Firm L	Invest	$(v_L^{(1,1)} - I, v_F^{(1,1)} - I)$	$(v_L^{(1,0)} - I, v_F^{(1,0)})$
	Wait	$(v_L^{(0,1)}, v_F^{(0,1)} - I)$	$(v_L^{(0,0)}, v_F^{(0,0)})$

FMA: dependence on risk aversion.

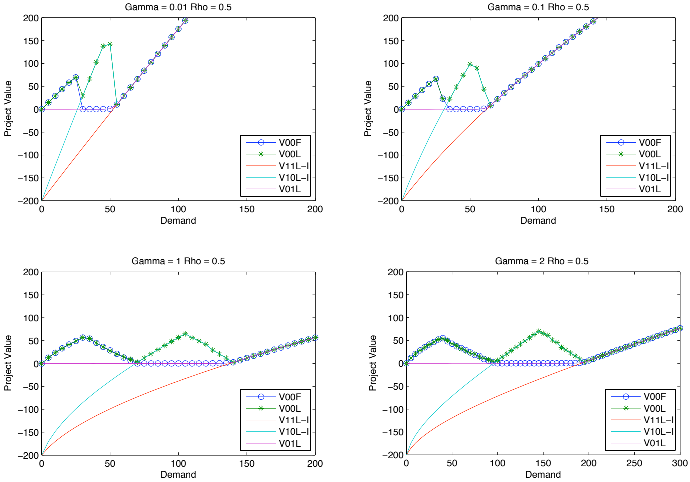

Figure: Project values in FMA case for different risk aversions.

FMA: dependence on correlation.

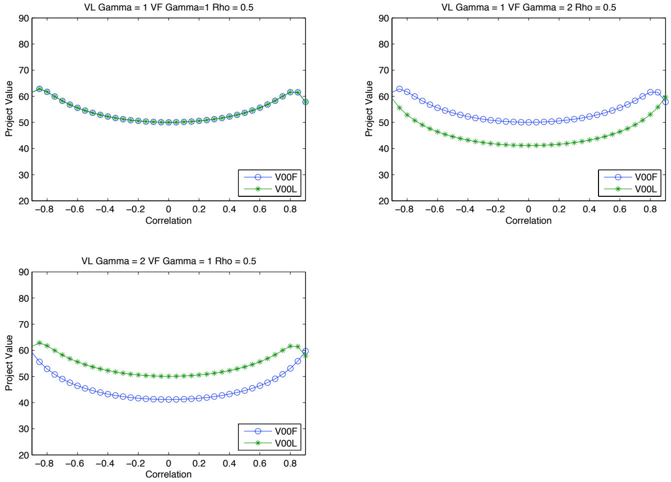

Figure: Project values in FMA case as function of correlation.

SMA: dependence on risk aversion

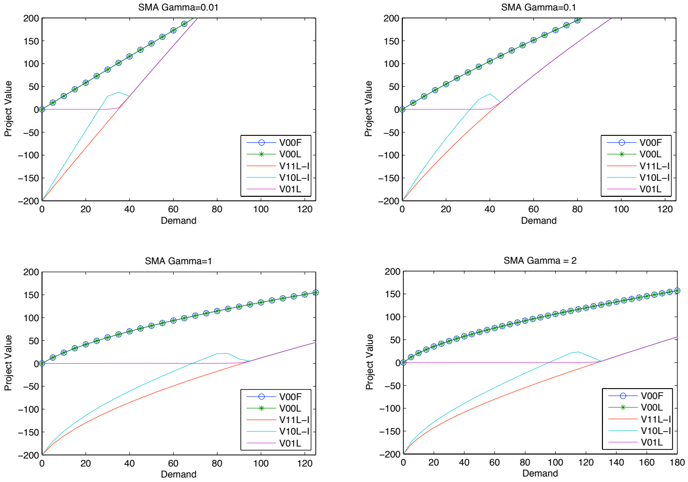

Figure: Project values in SMA case for different risk aversions.

SMA: dependence on correlation.

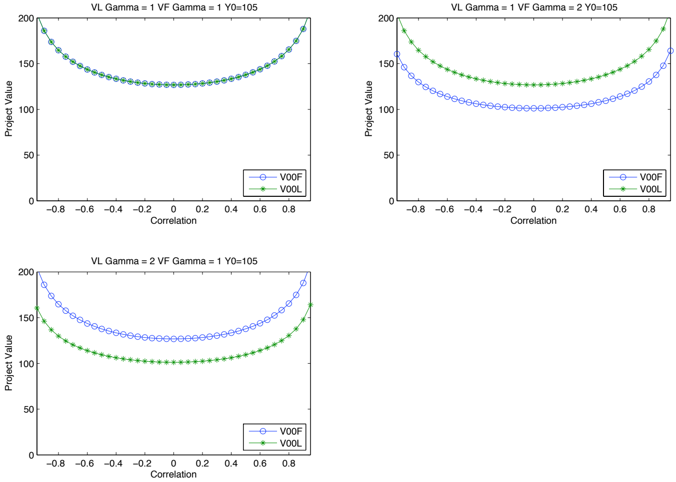

Figure: Project values in SMA case as function of correlation.

SMA x FMA

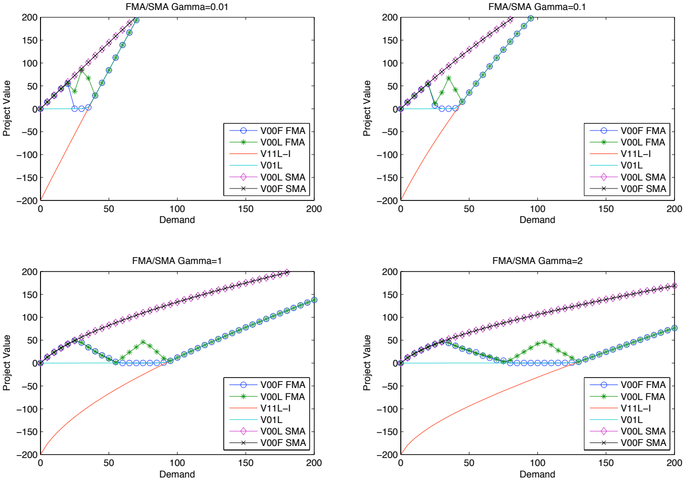

Figure: Project values for FMA and SMA.