The Reflected BSDE approach to Real Options in Incomplete markets

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Pisa, December 15, 2008

Successes and Limitations of Real Options

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- This is due to familiarity with the option valuation paradigm in financial markets and its lessons.
- But most of the literature in Real Options is based on one or both of the following assumptions: (1) infinite time horizon and (2) a perfectly correlated spanning asset.
- Though some problems have long time horizons (30 years or more), most strategic decisions involve much shorter times.
- The vast majority of underlying projects are not perfectly correlated to any asset traded in financial markets.

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- Instead, one needs to rely on risk preferences.
- The most widespread way to do this in the strategic decision making literature is to introduce a risk adjusted discount factor, which replaces the risk–free rate, and use dynamic programming.
- This approach lacks the intuitive understanding of opportunities as options.

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- For finite time horizons, a different version of the problem was solved Porchet, Touzi and Warin (2008) using the reflected BSDEs approach introduced in complete markets by Hamadène and Jeanblanc (2007).

A gentle introduction to BSDEs in Finance

- Given a terminal random variable $\xi \in \mathcal{F}_T$ and a generator function $f(t, y, z)$, a solution of a backward SDE is a pair of adapted processes (Y, Z) satisfying

$$Y_t = \xi - \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s' dW_s, \tag{1}$$

or equivalently

$$dY_t = f(t, Y_t, Z_t)dt + Z_t' dW_t \tag{2}$$

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- **Theorem** (Pardoux/Peng 1990): If ξ is square-integrable and f is uniformly Lipschitz, then the BSDE has a unique square-integrable solution.

First example: pricing and hedging in a complete market

- Consider the market

$$dB_t = B_t r_r dt, \tag{4}$$

$$dS_t^i = S_t^i \left[\mu_t dt + \sum_{j=1}^{n} \sigma_t^{ij} dW_t^j \right] \tag{5}$$

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- Given a claim $\xi \geq 0$, we look for a portfolio (V, π) satisfying

$$dX_t = r_t X_t dt + \pi_t' \sigma (dW_t + \lambda_t dt) \tag{6}$$

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- We see that this corresponds to a linear BSDE with

$$Y_t = X_t \tag{8}$$

$$Z_t = \sigma' \pi_t \tag{9}$$

$$f(t, Y_t, Z_t) = r Y_t + \lambda_t' Z_t \tag{10}$$

The Markovian Case

- For given (t, x), let $S_s^{t,x}$ be the solution of the forward SDE

$$S_s = x + \int_t^s \mu(u, S_u)du + \int_t^s \sigma(u, S_u)dWu, \quad t \leq s \leq T \quad (11)$$

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- When the coefficients satisfy certain Lipschitz and growth conditions, it can be shown that the solution can be written as $Y_s^{t,x} = u(s, S^{t,x})$ and $Z_s^{t,x} = \sigma' v(s, S_s^{t,x})$ for deterministic Borel functions $u(\cdot, \cdot)$ and $v(\cdot, \cdot)$.

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- Under additional regularity conditions on f and Φ (such as uniform continuity in x), it can be shown that the function $u(t, x) = Y_t^{t,x}$ is a viscosity solution of the PDE

$$u_t + \mathcal{L}u - f(t, x, u, \sigma' u_x) = 0, \quad (13)$$

where $\mathcal{L}$ is the generator of S_t.

Second example: utility maximization

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$$dS_t^i = S_t^i \left[\mu_t^i dt + \sum_{j=1}^{n} \sigma_t^{ij} dW_t^j \right], \quad i = 1, \ldots, d \leq n. \tag{14}$$

where μ_t^i, σ_t^{ij} are predictable uniformly bounded, σ_t is uniformly elliptic and let λ_t be a solution of

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- We are then interested in the optimization problem

$$u(x) := \sup_{\pi \in \mathcal{A}} E\left[-e^{-\gamma(X_T^\pi + B)} \right] \tag{17}$$

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  - $R_T^\pi = U(X_T^\pi + B)$
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$$R_t^\pi := -e^{-\gamma(X_t^\pi + Y_t^B)}, \tag{18}$$

- Here (Y^B, Z) is a solution of the BSDE

$$Y_t^B = B - \int_t^T f(s, Z_s)ds - \int_t^T Z_s' dW_s, \tag{19}$$

for a function f to be determined.

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$$= -e^{\gamma(x-Y_0)} e^{-\gamma\int_0^t (\pi_s'\sigma_s + Z_s')dW - \frac{1}{2}\int_0^t \gamma^2\|\pi_s'\sigma_s + Z_s'\|^2 ds} e^{\int_0^t v(s,\pi_s,Z_s)ds},$$

where $v(t,\pi,z) = -\gamma\pi'\sigma_t\lambda_t - \gamma f(t,z) + \frac{1}{2}\gamma^2\|\pi'\sigma_t + z'\|^2$.

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  where $v(t,\pi,z) = -\gamma\pi'\sigma_t\lambda_t - \gamma f(t,z) + \frac{1}{2}\gamma^2\|\pi'\sigma_t + z'\|^2$.
- We therefore seek for f such that $v(t,\pi_t,Z_t) \geq 0$ for all $\pi_t \in \mathcal{A}$ and $v(t,\pi_t^*,Z_t) = 0$ for some $\pi_t^* \in \mathcal{A}$.

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- Rearranging terms in v, we see that it suffices to take

$$f(t,z) = z\lambda_t - \frac{1}{2\gamma}\|\lambda_t\|^2 \quad (20)$$

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- This can be extended for the case of constrained portfolios.

Reflected BSDEs

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  3. A_t is continuous, increasing, $A_0 = 0$, and $\int_0^T (Y_t - C_t) dA_t = 0$.
- **Proposition** (El Karoui et al - 1997): Under further square–integrability conditions on (Y_t, Z_t, A_t) we have that

$$Y_t = \operatorname*{ess\,sup}_{\tau} E\left[-\int_t^{\tau} f(s, Y_s, Z_s)ds + C_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}} | \mathcal{F}_t \right]$$

The obstacle problem for PDEs

- Consider again the solution $S_s^{t,x}$ for the forward SDE (11) and let

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- Then, under certain continuity, integrability and growth conditions for Φ, g, f, it can be shown that the function $u(t,x) = Y_t^{t,x}$ is a viscosity solution of the obstacle problem

$$\min[-u_t - \mathcal{L}u - f(t,x,u,\sigma' u_x), u(t,x) - h(t,x)] = 0$$
$$u(T,x) = \Phi(x)$$

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- We can see that this corresponds to a reflected BSDE with

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- Moreover, setting $u(t, S_t) = e^{-rt}P_t$, we have that

$$\max[u_t + \mathcal{L}u, e^{-rt}(K - x)^+ - u(t,x)] = 0$$
$$u(T,x) = e^{-rT}(K - S_T)^+$$

The option to invest in an incomplete market

- Again let $r_t = 0$ and a two–factor model where discounted prices are given by

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dS_t &= \mu_1 S_t dt + \sigma_1 S_t dW_t^1 \\
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- We then model investment in the project as an American call option on V with strike price equals to the sunk cost, which is assumed to grow at rate r_t for simplicity.

Preferences

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- The indifference price for the option to invest in the project is the value p satisfying
$$u^0(x) = u(x-p, v)$$

System of reflected BSDEs

- From our previous example $u^0(x) = -e^{-\gamma(x+Y_0^1)}$ where

$$Y_t^1 = -\int_t^T f^1(Z_t^1)dt - \int_t^T Z_t^1 \cdot dW_t, \qquad (22)$$

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- Similarly, we will show that $u(x, v) = -e^{-\gamma(x+Y_0^2)}$ where

$$Y_t^2 = (V_T - I)^+ - \int_t^T f^2(Z_t^2)dt - \int_t^T Z_t^2 \cdot dW_t + (A_T - A_t)$$
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for $f^2(z_1, z_2) = \frac{\gamma}{2}\left(\frac{\lambda_2}{\gamma} - z_2\right)^2 + z \cdot \lambda - \frac{\|\lambda\|^2}{2\gamma}$.

Sketch of the proof

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$$\mathbb{E}\left[-e^{-\gamma\left(X_\tau^\pi + (V_\tau - I)^+ + \int_\tau^T \bar{\pi}\frac{dS}{S}\right)}\right] \leq -e^{-\gamma\left(X_\tau^\pi + (V_\tau - I)^+ + Y_\tau^1\right)}$$

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- We obtain equalities by setting

$$\tau^* = \inf\{0 \leq t \leq T : Y_t^2 = (V_t - I)^+ + Y_t^1\}$$

$$\pi_t^* \sigma = \begin{cases} \lambda_1/\gamma - Z_{1,t}^2 & 0 \leq t \leq \tau^* \\ \lambda_1/\gamma - Z_{1,t}^1 & \tau < t \leq T \end{cases}$$

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  where $f(z_1, z_2) = z_1 \lambda_1 + \frac{\gamma}{2}(z_2)^2$
- We can then characterize the indifference price as the initial value of the viscosity solution of an obstacle problem and calculate it numerically.

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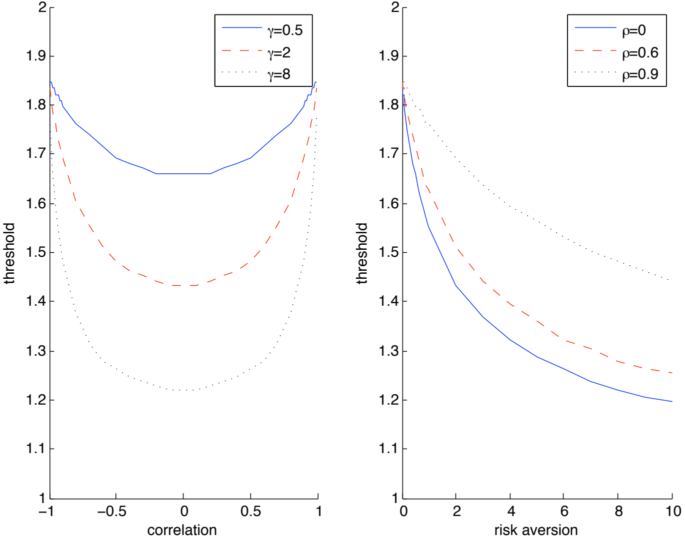
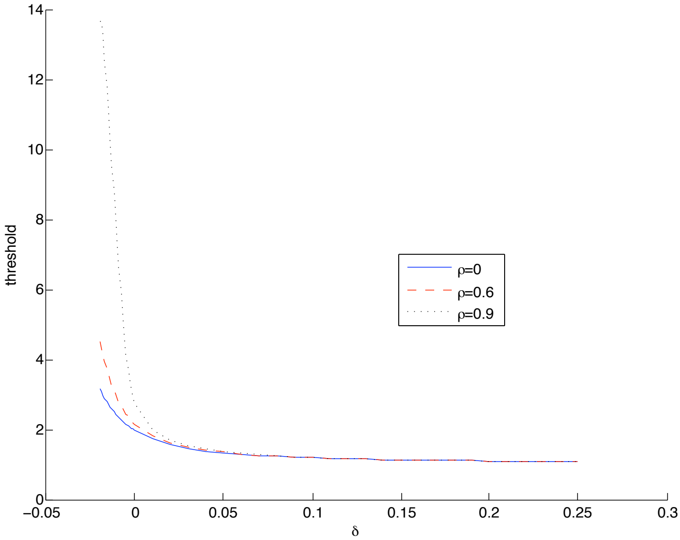
- p is an increasing function of σ_2 for $\delta > 0$, but it is decreasing in σ_2 when $\delta < 0$.

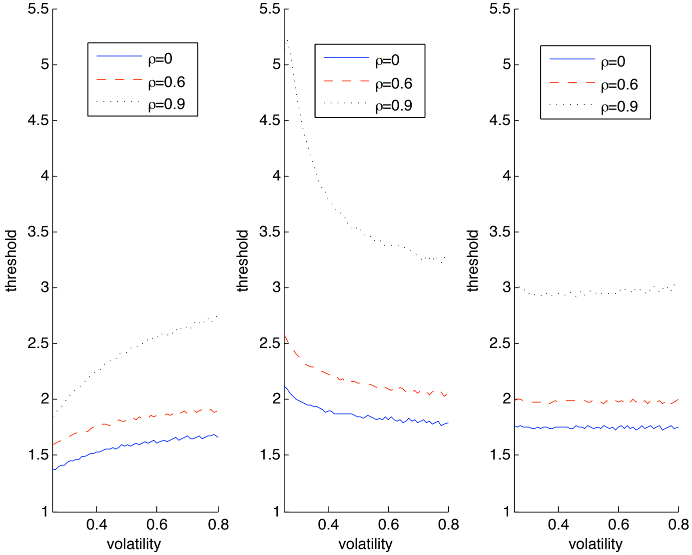
Dependence with Correlation and Risk Aversion



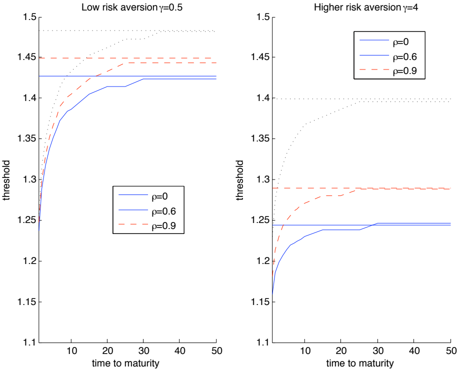
Dependence with Dividend Rate



Dependence with Volatility



Dependence with Time to Maturity



Depreciation

- Instead of the project value itself, we can model the output cash–flow rate

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- If the project expires at an exponentially distributed time τ, then

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- How can we value the combine entry/exit options ?

Investment strategies and stopping times

- An entry/exit strategy in this setting is a process

$$\xi_t = \sum_{n \geq 1} \mathbf{1}_{\{\tau_{2n-1} \leq t < \tau_{2n}\}}$$

where $\tau_0 = 0$, τ_{2n-1} are investment times and τ_{2n} are abandonment time.

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- For a given ξ, we consider the wealth process

$$\begin{aligned} dX_t^{\pi,\xi} &= \pi_t \sigma (dW_t^1 + \lambda_1 dt), \quad \tau_k \leq t < \tau_{k+1} \\ X_{\tau_{2n-1}}^{\pi,\xi} &= X_{\tau_{2n-1}^-}^{\pi,\xi} + V(P_{\tau_{2n-1}}) - I \\ X_{\tau_{2n}}^{\pi,\xi} &= X_{\tau_{2n}^-}^{\pi,\xi} - E \end{aligned}$$

Utility valuation

- We can then show that

$$u(t, x, P) = \sup_{\pi,\xi} E\left[-e^{-\gamma X^{\pi,\xi}} | X_t^{\pi,\xi} = x\right] = -e^{x+Y_0^2},$$

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- Here Y_0^2 is the solution of the following system of reflected BSDE

$$Y_t^1 = \max(V_T, -E) - \int_t^T f^1(Z_t^1)dt - \int_t^T Z_t^1 \cdot dW_t + (A_T^1 - A_t^1)$$

$$Y_t^2 = \max(V_T - I, 0) - \int_t^T f^2(Z_t^2)dt - \int_t^T Z_t^2 \cdot dW_t + (A_T^2 - A_t^2)$$

$$Y_t^2 \geq Y_t^1 + (V(P_t) - I)^+, \qquad Y_t^1 \geq Y_t^2 - E$$

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