Indifference prices for general semimartingales

M. R. Grasselli
Sharcnet Chair in Financial Mathematics

Department of Mathematics and Statistics
McMaster University

Joint work with M. Frittelli and S. Biagini

Stevanovich Center for Financial Mathematics
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The Hedging Problem

- Consider the following stochastic optimization problem:

$$U_B^W(x) = \sup_{H \in \mathcal{H}^W} E\left[u\left(x + \int_0^T HdS - B\right)\right] \qquad (1)$$

- The utility $u : \mathbb{R} \to \mathbb{R}$ is *strictly concave*, *increasing*, *differentiable* and satisfies the *Inada conditions*

$$\lim_{x \to -\infty} u'(x) = +\infty \quad \text{and} \quad \lim_{x \to +\infty} u'(x) = 0. \qquad (2)$$

- The initial constant endowment is $x \in \mathbb{R}$ and the fixed time horizon is $T \in (0, +\infty]$.
- The underlying process S is an $\mathbb{R}^d-$valued càdlàg semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, which is not assumed to be locally bounded.
- B is an $\mathcal{F}_T-$measurable liability satisfying appropriate integrability conditions.

Admissible integrands, suitability and compatibility

- Given $W \in L^0_+$, define the W–admissible strategies as

$$\mathcal{H}^W := \left\{ H \in L(S) : \int_0^t HdS \geq -cW, \text{ for some } c > 0 \right\}.$$

- We say that $W \geq 1$ is suitable for S if for each $i = 1, \dots, d$, there exists a process $H^i \in L(S^i)$ such that

$$P(\{\omega \mid \exists t \geq 0 \text{ such that } H^i_t(\omega) = 0\}) = 0 \tag{3}$$

and

$$\left| \int_0^t H^i dS^i \right| \leq W, \quad \forall t \in [0, T]. \tag{4}$$

- We say that $W \in L^0_+$ is compatible with the utility function u if

$$E[u(-\alpha W)] > -\infty \text{ for all } \alpha > 0 \tag{5}$$

and that it is weakly compatible with u if

$$E[u(-\alpha W)] > -\infty \text{ for some } \alpha > 0. \tag{6}$$

Terminal values and duality

- Given a suitable and compatible random variable W, define

$$K^W = \left\{ \int_0^T HdS : H \in \mathcal{H}^W \right\} \tag{7}$$

so that the primal problem (1) becomes:

$$\sup_{k \in K^W} E[u(x + k - B)]. \tag{8}$$

- We then want to define an appropriate cone C^W, related to K^W, and invoke Fenchel's duality theorem.
- For this, we need to choose a Banach spaces and its topological dual in order to define the polar set $(C^W)^0$.
- Classically, the spaces (L^∞, ba) were successfully used when dealing with locally bounded traded assets. In order to accommodate more general markets and inspired by the compatibility conditions above, we argue instead for the use of an appropriate Orlicz space and its dual.

Orlicz spaces

- Consider the Young function $\widehat{u}: \mathbb{R} \to [0, +\infty)$ associated with the utility function u, defined as

$$\widehat{u}(x) = -u(-|x|) + u(0).$$

- Its corresponding Orlicz space is

$$L^{\widehat{u}}(P) = \{f \in L^0(P) : E[\widehat{u}(\alpha f)] < +\infty \text{ for some } \alpha > 0\},$$

equipped with the Luxemburg norm

$$\|f\|_{\widehat{u}} = \inf\left\{c > 0 : E\left[\widehat{u}\left(\frac{f}{c}\right)\right] \leq 1\right\}.$$

- We then have $L^\infty \subseteq L^{\widehat{u}}(P) \subseteq L^1(P)$
- Next we consider the closed subspace

$$M^{\widehat{u}}(P) = \{f \in L^{\widehat{u}}(P) : E[\widehat{u}(\alpha f)] < +\infty \text{ for all } \alpha > 0\}.$$

- In general $M^{\widehat{u}} \subset L^{\widehat{u}}$ (strict inclusion).

Compatibility revisited

- The Young function $\widehat{u}$ carries information about the utility on large losses, in the sense for $\alpha > 0$ we have that

$$E[\widehat{u}(\alpha f)] < +\infty \qquad \Longleftrightarrow \qquad E[u(-\alpha|f|)] > -\infty. \tag{9}$$

- We can then see that a positive random variable W is compatible (resp. weakly compatible) with the utility function u if and only if $W \in M^{\widehat{u}}$ (resp. $W \in L^{\widehat{u}}$).

Complementary spaces

- The convex conjugate of $\widehat{u}$, called the complementary Young function in the theory Orlicz spaces, is
$$\widehat{\Phi}(y) := \sup_x \{x|y| - \widehat{u}(x)\} = \Phi(|y| + \beta) - \Phi(\beta),$$
where $\beta = u'(0) > 0$ and Φ is the concave conjugate of u.
- We consider the Orlicz space $L^{\widehat{\Phi}}$ endowed with the Orlicz norm
$$|||g|||_{\widehat{\Phi}} = \sup\{|E[fg]|, f \in L^{\widehat{u}}, E[\widehat{u}(f)] \leq 1\}.$$
- It then follows that $(M^{\widehat{u}})^* = L^{\widehat{\Phi}}$ in the sense that if $z \in (M^{\widehat{u}})^*$ is a continuous linear functional on $M^{\widehat{u}}$, then there exists a unique $g \in L^{\widehat{\Phi}}$ such that
$$z(f) = \int_\Omega fgdP, \qquad f \in M^{\widehat{u}},$$
with $\|z\|_{(M^{\widehat{u}})^*} := \sup_{\|f\|_{\widehat{u}} \leq 1} |z(f)| = |||g|||_{\widehat{\Phi}}$.

The dual of $L^{\widehat{u}}$

- It follows from the properties of the pair $(\widehat{u}, \widehat{\Phi})$ that each element $z \in (L^{\widehat{u}})^*$ can be uniquely expressed as

$$z = z^r + z^s,$$

where the *regular* part z^r is given by

$$z^r(f) = \int_\Omega fgdP, \qquad f \in L^{\widehat{u}},$$

for a unique $g \in L^{\widehat{\Phi}}$, and the *singular* part z^s satisfies

$$z^s(f) = 0, \qquad \forall f \in M^{\widehat{u}}. \tag{10}$$

- That is, $(L^{\widehat{u}})^* = (M^{\widehat{u}})^* \oplus (M^{\widehat{u}})^\perp$.

Positive singular functionals

- Consider the concave integral functional

$$\begin{aligned} I_u : L^{\widehat{u}} &\rightarrow [-\infty, \infty) \\ f &\mapsto E[u(f)] \end{aligned}$$

  with effective domain

$$\mathcal{D}(P) = \left\{ f \in L^{\widehat{u}}(P) \mid E[u(f)] > -\infty \right\}. \qquad (11)$$

- One consequence of choosing the correct Orlicz spaces is that the norm of a *non negative* singular element $0 \leq z \in (M^{\widehat{u}})^{\perp}$ satisfies

$$\|z\|_{(L^{\widehat{u}}(P))^*} := \sup_{\|f\|_{\widehat{u}} \leq 1} |z(f)| = \sup_{f \in \mathcal{D}(P)} z(-f),$$

Dual Variables

- Given a loss variable $W \in \mathbb{S} \cap L^{\widehat{u}}$ we define the cone

$$C^W = (K^W - L^0_+) \cap L^{\widehat{u}}.$$

- Define the polar cone

$$(C^W)^0 := \left\{ z \in (L^{\widehat{u}})^* \mid z(f) \leq 0, \quad \forall f \in C^W \right\}, \tag{12}$$

which satisfies $(C^W)^0 \subseteq (L^{\widehat{u}})^*_+$, since $(-L^{\widehat{u}}_+) \subseteq C^W$.

- The subset of normalized functionals in $(C^W)^0$ is defined by

$$\mathcal{M}^W := \{ Q \in (C^W)^0 \mid Q(\mathbf{1}_\Omega) = 1 \}. \tag{13}$$

- It was shown in Biagini-Frittelli (2006) that

$$\mathcal{M}^W \cap L^1(P) = \mathbb{M}_\sigma \cap L^{\widehat{\Phi}}. \tag{14}$$

Conditions of the claim

- For the main duality result, we consider claims in the set $\mathcal{A}_u$ characterized by the conditions claims $B \in \mathcal{F}_T$ satisfying

$$\begin{aligned} E[u(f-B)] &< +\infty, \qquad \forall f \in L^{\widehat{u}} \\ E[u(-(1+\varepsilon)B^+)] &> -\infty, \text{ for some } \varepsilon > 0 \end{aligned}$$

- Observe that these conditions do not capture the risk in B^-, which are *gains* for the seller.
- For the domain of the indifference price price, we consider the set $\mathcal{B} := \mathcal{A}_u \cap L^{\widehat{u}}$, that is, claims satisfying

$$E[u(-(1+\varepsilon)B^+)] > -\infty, \quad E[u(-\varepsilon B^-)] > -\infty, \text{ for some } \varepsilon > 0.$$

Extended functionals

- Observe that $B \in \mathcal{A}_u$ does not necessarily imply that $B \in L^{\widehat{u}}$.
- Accordingly, for any $Q \in (L^{\widehat{u}})^*_+$ we define $\widehat{Q} : L^{\widehat{u}}_{neg} \to \mathbb{R} \cup \{+\infty\}$ by

$$\widehat{Q}(g) := \sup \left\{ Q(f) \mid f \in L^{\widehat{u}} \text{ and } f \leq g \right\}, \tag{15}$$

  where

$$L^{\widehat{u}}_{neg} := \left\{ f \in L^0 \mid f^- \in L^{\widehat{u}} \right\}.$$

- Then $\widehat{Q}$ is a well–defined, positively homogeneous, additive, monotone extension of Q.

Extended domains

- The dual objective function has a term $E\left[\Phi\left(\lambda\frac{dQ^r}{dP}\right)\right]$.
- This leads us to consider the set

$$\mathcal{L}_\Phi := \left\{Q \text{ probab }, Q \ll P \mid E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right] < \infty \text{ for some } \lambda > 0\right\}$$

- Note that $\mathcal{L}_\Phi = L_+^{\widehat{\Phi}}$ whenever $\Phi(0) < \infty$ (i.e $u(\infty) < \infty$)
- For $B \in \mathcal{A}_u$, define

$$\begin{aligned} \mathcal{N}_B^W &:= \{Q \in \mathcal{M}^W \mid Q^r \in \mathcal{L}_\Phi, \widehat{Q}(-B) \in \mathbb{R}\} \\ K_B^W &:= \{f \in L^0 \mid E_{Q^r}[f] \le \widehat{Q^s}(-B) + \|Q^s\|, \forall Q \in \mathcal{N}_B^W\} \end{aligned}$$

- If $\mathcal{N}_B^W \neq 0$, we have that $K^W \in K_B^W$

Main duality result

Theorem (BFG)

Suppose that $B \in \mathcal{A}_u$ and that there exists $W \in \mathbb{S} \cap L^{\widehat{u}}$ satisfying

$$\sup_{H \in \mathcal{H}^W} E\left[u\left(\int_0^T HdS - B\right)\right] < u(\infty). \tag{16}$$

Then $\mathcal{N}_B^W \neq 0$ and

$$\begin{aligned} \sup_{H \in \mathcal{H}^W} & E\left[u\left(\int_0^T HdS - B\right)\right] \\ &= \min_{\lambda > 0,\ Q \in \mathcal{N}_B^W} \left\{\lambda \widehat{Q}(-B) + E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] + \lambda \|Q^s\|\right\}, \end{aligned}$$

If $B \in M^{\widehat{u}}$, then $Q^s(B) = 0$. Moreover, if $W \in M^{\widehat{u}}$ and $B \in M^{\widehat{u}}$ then $\mathcal{N}_B^W$ can be replaced by $\mathbb{M}_\sigma \cap L^{\widehat{\Phi}}$ and no singular term appears.

The solution to the primal problem

- We say that u satisfies Assumption (A) if

$$\mathcal{L}_\Phi = \left\{ Q \text{ probab }, Q \ll P \mid E\left[\Phi\left(\lambda \frac{dQ}{dP}\right)\right] < \infty \text{ for all } \lambda > 0 \right\}$$

- In this case, we have that

$$U_B^W = E[u(f_B - B)], \tag{17}$$

where the maximizer is

$$f_B = \left(-\Phi'(\lambda_B \frac{dQ_B^r}{dP}) + B\right) \in K_B^W \tag{18}$$

and satisfies

$$E_{Q_B^r}[f_B] = \widehat{Q_B^s}(-B) + \|Q_B^s\|. \tag{19}$$

The indifference price and its domain

- Following Hodges and Neuberger (1989), we define the *indifference price* $\pi(B)$ for the seller of a claim B as the the implicit solution of the equation

$$\sup_{H\in\mathcal{H}^W} E\left[u\left(x+\int_0^T HdS\right)\right] = \sup_{H\in\mathcal{H}^W} E[u(x+\pi(B) + \int_0^T HdS - B)]. \tag{20}$$

- We compute indifference prices for claims in $\mathcal{B} := \mathcal{A}_u \cap L^{\widehat{u}}$.
- We have that $\mathcal{B} = \{B \in L^{\widehat{u}} \mid (-B) \in \operatorname{int}(\operatorname{Dom}(I_u))\}$ and has the following properties
    1. $\mathcal{B}$ is convex, open and contains $M^{\widehat{u}}$
    2. $B_1 \in \mathcal{B}$ and $B_2 \leq B_1$ implies $B_2 \in \mathcal{B}$
    3. $B \in \mathcal{B}$ and $C \in M^{\widehat{u}}$ implies that $B + C \in \mathcal{B}$

Properties of the indifference price

The indifference price $\pi : \mathcal{B} \to \mathbb{R}$ is well defined, convex, monotone, translation invariant, norm continuous, and admits the representation:

$$\pi(B) = \max_{Q \in \mathcal{M}^W} \left[Q(B) - \alpha(Q) \right], \tag{21}$$

where the penalty term is given by:

$$\alpha(Q) := x + \|Q^s\| + \inf_{\lambda > 0} \left\{ \frac{E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] - U_0^W(x)}{\lambda} \right\}. \tag{22}$$

Volume asymptotics

Denoting by $\mathcal{Q}_0^W(x)$ the set of dual optimizers for the Merton problem, then fo any $B \in \mathcal{B}$ we have

$$\lim_{b \downarrow 0} \frac{\pi(bB)}{b} = \max_{Q \in \mathcal{Q}_0^W(x)} Q(B) \tag{23}$$

If B is in $M^{\widehat{u}}$,

$$\lim_{b \to +\infty} \frac{\pi(bB)}{b} = \sup_{Q \in \mathcal{N}^W} Q(B) \tag{24}$$

If $W \in M^{\widehat{u}}$ and $B \in M^{\widehat{u}}$, the two volume asymptotics above become

$$\lim_{b \downarrow 0} \frac{\pi(bB)}{b} = E_{Q^*}[B], \qquad \lim_{b \to +\infty} \frac{\pi(bB)}{b} = \sup_{Q \in M_\sigma \cap \mathbb{P}_\Phi} E_Q[B]$$

where the probability $Q^* \in M_\sigma \cap \mathbb{P}_\Phi$ is the unique dual minimizer in $\mathcal{Q}_0^W(x)$.

Further work

- Levy market example
- Utilities on a half line
- Risk measures on Orlicz spaces