Closed form valuation of volatility claims

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1. Introduction

- **Optimal hedging portfolio:** the strategy followed by an investor who, when faced with a financial liability B maturing at a future time T, tries to solve the stochastic control problem

$$u(x) = \sup_{H \in \mathcal{A}} E\left[U\left(X_T - B\right) | X_0 = x\right], \qquad (1)$$

  where X_T is the discounted terminal wealth obtained when trading on assets with discounted prices $S_t = (S_t^1, \ldots, S_t^d)$ according to an allocation process $H_t = (H_t^1, \ldots, H_t^d)$

- **Utility function:** $U : \mathbf{R} \to \mathbf{R} \cup \{-\infty\}$, assumed to be concave, strictly increasing and differentiable function.

- **Domain of optimization:** self–financing portfolios, that is, wealth processes evolving according to the stochastic differential equation

$$\begin{cases} dX_t = H_t dS_t \\ X_0 = x, \end{cases} \tag{2}$$

or equivalently, wealth processes given by

$$X_t = x + \int_0^t H_u dS_u, \quad t \in [0, T], \tag{3}$$

where S_t is an $\mathbf{R}^d$–valued semimartingale on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ and H_t an $\mathbf{R}^d$–valued predictable S–integrable. Further restrictions on the class $\mathcal{A}$ of admissible portfolios appearing in the domain of optimization are imposed by economic reasoning.

- **Market Model:** We consider two factor stochastic volatility models of the form

$$d\bar{S}_t = \bar{S}_t[\mu(t, Y_t)dt + \sigma(t, Y_t)dW_t^1]$$
$$dY_t = a(t, Y_t)dt + b(t, Y_t)[\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2] \quad (4)$$

with initial values $\bar{S}_0, Y_0 \geq 0$, for deterministic functions μ, a, b and independent one dimensional P–Brownian motions W_t^1 and W_t^2 with constant correlation $|\rho| \leq 1$. We introduce a riskless bank account S_t^0 initialized at 1 and governed by

$$dS_t^0 = r_t S_t^0 dt. \quad (5)$$

The *discounted* price of the risky asset is $S_t = \bar{S}_t / S_t^0$. It follows from Ito's formula that

$$dS_t = S_t[(\mu(t, Y_t) - r_t)dt + \sigma(t, Y_t)dW_t^1]. \tag{6}$$

Using the self–financing condition (2), we immediately obtain that the wealth process satisfies

$$dX_t = H_t S_t[(\mu(t, Y_t) - r)dt + \sigma(t, Y_t)dW_t^1], \tag{7}$$

where we have taken $r_t = r$ to be constant for simplicity.

2. Utility based pricing

For Markovian markets such as (4) and claims of the form $B = B(S_T, Y_T)$ consider the larger class of optimization problems defined by

$$u(t,x,s,y) = \sup_{H \in \mathcal{A}_t} E_{t,s,y}[U(X_T - B(S_T, Y_T))|X_t = x], \qquad (8)$$

for $t \in (0,T)$, where $x \in \mathbf{R}$ denotes some arbitrary level of wealth, $\mathcal{A}_t$ denotes admissible portfolios starting at time t and $E_{t,s,y}[\cdot]$ denotes expectation with respect to the joint probability law at time t of the processes S_u, Y_u satisfying

$$\begin{aligned} dS_u &= S_u[(\mu(u,Y_u) - r)du + \sigma(u,Y_u)dW_u^1], \qquad (9) \\ dY_u &= a(u,Y_u)du + b(u,Y_u)[\rho dW_u^1 + \sqrt{1-\rho^2}dW_u^2], \end{aligned}$$

for $u \geq t$, with initial condition $S_t = s$ and $Y_t = y$.

Now suppose that (8) has an optimizer H^B, that is, assume that

$$u(t,x,s,y) = E_{t,s,y}[U(x + (H^B \cdot S)_t^T - B(S_T, Y_T))].$$

Define the *certainty equivalent* $c_t^B = c^B(t,x,s,y)$ for the claim B at time t via the equation

$$U(x - c_t^B)) = E_{t,s,y}[U(x + (H^B \cdot S)_t^T - B(S_T, Y_T))]. \tag{10}$$

If we set $B = 0$, then we obtain Merton's optimal investment problem and denote the certainty equivalent by $c_t^0 = c^0(t,x,s,y)$. Notice that

$$\begin{aligned} -c_t^0 \geq 0 \iff U(x) &\leq U(x - c_t^0) \\ &= E_{t,s,y}[U(x + (H^0 \cdot S)_t^T)]. \end{aligned}$$

Now consider an investor with utility U who at time $t \in (0, T)$ has wealth x and ponders the possibility of charging a premium for issuing a liability B maturing at T. The *indifference price* for the claim B is defined to be the premium that makes the investor indifferent between making the deal or not, that is, the unique solution $\pi^B = \pi^B(t, x, s, y)$ (if it exists) to the equation

$$\sup_{H \in \mathcal{A}_t} E_{t,s,y}[U(x+(H\cdot S)_t^T] = \sup_{H \in \mathcal{A}_t} E_{t,s,y}[U(x+\pi^B+(H\cdot S)_t^T - B(S_T, Y_T)].$$

From the definition of the certainty equivalent, we see that this equation is equivalent to

$$U(x - c_t^0) = U(x + \pi^B - c_t^B), \tag{11}$$

so that the indifference price is given by

$$\pi^B = c^B(t, x + \pi^B, s, y) - c^0(t, x, s, y). \tag{12}$$

From now on, let us concentrate on exponential utilities of the form

$$U(x) = -e^{-\gamma x}, \tag{13}$$

where $\gamma > 0$ is the risk–aversion parameter. We can factorize the value function $u(t, x, s, y)$ in (8) as

$$\begin{aligned} u(t, x, s, y) &= \sup_{H \in \mathcal{A}_t} E_{t,z}\left[-e^{-\gamma\left(x + (H \cdot S)_t^T - B(Z_T)\right)}\right] \\ &= -e^{-\gamma x} \inf_{H \in \mathcal{A}_t} E_{t,z}\left[e^{-\gamma\left((H \cdot S)_t^T - B(Z_T)\right)}\right] \\ &=: U(x) v(t, s, y). \end{aligned} \tag{14}$$

It follows that the certainty equivalent is wealth independent and given by

$$c^B(t, s, y) = \frac{1}{\gamma} \log v(t, s, y). \tag{15}$$

Analogously, setting $B = 0$ gives the certainty equivalent for the Merton problem with exponential utility as

$$c^0(t,s,y) = \frac{1}{\gamma} \log v^0(t,s,y), \tag{16}$$

where $v^0(t,s,y)$ is defined by

$$v^0(t,s,y) := \inf_{H \in \mathcal{A}_t} E_{t,s,y}\left[e^{-\gamma (H \cdot S)_t^T}\right]. \tag{17}$$

It is now immediate that the indifference price process for the claim B obtained from an exponential utility is given by

$$\pi^B(t,s,y) = c^B(t,s,y) - c^0(t,s,y) = \frac{1}{\gamma} \log \frac{v(t,s,y)}{v^0(t,s,y)}. \tag{18}$$

3. The PDE for the indifference price

By the dynamic programming principle, the value function $u(t,x,s,y)$ satisfies the Hamilton–Jacobi–Belmann equation

$$\frac{\partial u}{\partial t} + s^2\sigma^2 u_{ss} + b^2 u_{yy} + b\rho s\sigma u_{ys} + s(\mu - r)u_s + a u_y \tag{19}$$
$$+ \max_h \left\{ \frac{1}{2} h^2 s^2 \sigma^2 u_{xx} + + b\rho h s\sigma u_{xy} + h s^2 \sigma^2 u_{xs} + hs(\mu - r)u_x \right\} = 0$$

with boundary condition $u(T,z) = -e^{-\gamma(x - B(s,y))}$ and the optimal portfolio process is given by

$$H_t^B = h^B(t, X_t, S_t, Y_t) \tag{20}$$

where h^B is the optimizer of the expression above.

Direct substitution of (14) into this HJB problem transforms it into the search for a wealth independent function $v(t, s, y)$ satisfying

$$\frac{\partial v}{\partial t} + \frac{1}{2}\left(s^2\sigma^2 v_{ss} + 2b\rho s\sigma v_{ys} + b^2 v_{yy}\right) + s(\mu - r)v_s + av_y$$
$$\min_h \left\{\frac{\gamma^2}{2}h^2 s^2\sigma^2 v - \gamma hs[b\rho\sigma v_y + s\sigma^2 v_s + (\mu - r)v]\right\} = 0$$

with the boundary condition $v(T, s, y) = e^{-\gamma B(s,y)}$, where the minimizer is of the form $H_t^B = h^B(t, s, y)$.

The minimizer of this expression as a function is clearly given by

$$h^B(t, s, y) = \frac{1}{\gamma}\frac{v_s}{v} + \frac{b\rho}{\gamma s\sigma}\frac{v_y}{v} + \frac{(\mu - r)}{\gamma s\sigma^2}. \qquad (21)$$

The partial differential equation satisfied by the optimal function $v(t,s,y)$ is then

$$\frac{\partial v}{\partial t} + \frac{1}{2}\left(s^2\sigma^2 v_{ss} + 2b\rho s\sigma v_{ys} + b^2 v_{yy}\right) + \left[a - \frac{b\rho(\mu - r)}{\sigma}\right] v_y - \frac{1}{2}\left[\frac{1}{v}(b\rho v_y + s\sigma v_s)^2 + \frac{(\mu - r)^2}{\sigma^2} v\right] = 0, \tag{22}$$

subject to the boundary condition $v(T,s,y) = e^{\gamma B(s,y)}$.

From (15), we find that the certainty equivalent process $c^B(t,z)$ is a solution to the partial differential equation

$$\frac{\partial c^B}{\partial t} + \frac{1}{2}(s^2\sigma^2 c^B_{ss} + 2s\sigma b\rho\, c^B_{sy} + b^2 c^B_{yy}) + \left[a - \frac{b\rho(\mu - r)}{\sigma}\right] c^B_y - \frac{(\mu - r)^2}{2\gamma\sigma^2} + \frac{\gamma}{2} b^2(1 - \rho^2)(c^B_y)^2 = 0, \tag{23}$$

In terms of the certainty equivalent process, the optimal portfolio is expressed as

$$h^B(t,s,y) = c_s^B + \frac{b\rho}{s\sigma} c_y^B + \frac{(\mu - r)}{\gamma s \sigma^2}. \tag{24}$$

The partial differential equation satisfied by $c^0(t,z)$ is identical to (23), but with the boundary condition $c^0(T,s,y) = 0$, instead of $c^B(T,s,y) = B(s,y)$. From (18), we obtain that the indifference price $\pi(t,z)$ for the claim B with respect to the exponential utility is a solution to the equation

$$\begin{aligned} \frac{\partial \pi^B}{\partial t} &+ \frac{1}{2}(s^2\sigma^2\pi_{ss}^B + 2s\sigma b\rho\,\pi_{sy}^B + b^2\pi_{yy}^B) + \frac{\gamma}{2} b^2(1-\rho^2)(\pi_y^B)^2 \\ &+ \left[a - \frac{b\rho(\mu - r)}{\sigma} + \gamma b^2(1-\rho^2)c_y^0\right]\pi_y^B = 0. \end{aligned} \tag{25}$$

4. Volatility claims

The equations from the previous section simplify considerably in the case when the claim is independent of the process S_t. For pure volatility claims of the form $B = B(Y_T)$, the equation for the certainty equivalent $c_t^B = c^B(t,y)$ is reduced to

$$\frac{\partial c^B}{\partial t} + \left[a - \frac{b\rho(\mu - r)}{\sigma}\right] c_y^B + \frac{1}{2} b^2 c_{yy}^B - \frac{(\mu - r)^2}{2\gamma\sigma^2} + \frac{\gamma}{2} b^2 (1 - \rho^2)(c_y^B)^2 = 0, \tag{26}$$

subject to the boundary condition $c^B(T,y) = B(y)$, whereas the optimal hedging portfolio is given by

$$h^B(t,s,y) = \frac{1}{s}\left[\frac{b\rho}{\sigma} c_y^B + \frac{\mu}{\gamma\sigma^2}\right]. \tag{27}$$

We can now use the transformation

$$c^B(t,y) = \frac{\delta}{\gamma} \log f(t,y), \tag{28}$$

for a constant δ still to be determined. Substitution into (26) leads to

$$\begin{aligned} \frac{\partial f}{\partial t} &+ \frac{1}{2} b^2 f_{yy} + \left[a - \frac{b\rho(\mu - r)}{\sigma} \right] f_y - \frac{(1-\rho^2)(\mu - r)^2}{2\sigma^2} f \\ &+ \frac{1}{2} b^2 [(1-\rho^2)\delta - 1](f_y)^2 = 0. \end{aligned} \tag{29}$$

Therefore, if we set

$$\delta = (1-\rho^2)^{-1}, \tag{30}$$

we obtain that $f(t,y)$ must solve the linear parabolic final value problem

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{1}{2} b^2 f_{yy} + \left[a - \frac{b\rho(\mu - r)}{\sigma} \right] f_y - \frac{(1-\rho^2)(\mu - r)^2}{2\sigma^2} f = 0 \\ f(T,y) = e^{\gamma(1-\rho^2)B(y)} \end{aligned} .$$

Under the appropriate growth and boundedness assumptions on the coefficient functions μ, σ, a and b, we can use the Feynman–Kac formula to represent the solution to the problem above as

$$f(t,y) = E^Q_{t,y}\left[e^{-\int_t^T R(s,Y_s)ds} e^{\gamma(1-\rho^2)B(Y_T)}\right], \tag{31}$$

where we define

$$R(t,y) = \frac{(1-\rho^2)(\mu(t,y)-r)^2}{2\sigma(t,y)^2} \tag{32}$$

and $E^Q_{t,y}[\cdot]$ denotes the expectation with respect to the probability law at time $s=t$ of the solution to

$$\begin{aligned} dY_s &= \left[a - \frac{b(\mu - r)\rho}{\sigma}\right] ds + b\left[\rho d\widetilde{W}^1_s + \sqrt{1-\rho^2} d\widetilde{W}^2_s\right], \\ Y_t &= y \end{aligned} \tag{33}$$

for a pair of independent one dimensional Q–Brownian motions $\widetilde{W}^1_t, \widetilde{W}^2_t$, for a probability measure Q on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]})$.

Comparison with the original SDE (4) for the non-traded asset leads to the identification

$$\begin{aligned} d\widetilde{W}_t^1 &= dW_t^1 + \lambda_t^1 dt \\ d\widetilde{W}_t^2 &= dW_t^2 + \lambda_t^2 dt, \end{aligned}$$

where the "market price of risk" vector process $\lambda_t = (\lambda_t^1, \lambda_t^2)$ is constrained by

$$\rho \lambda_t^1 + \sqrt{1-\rho^2}\lambda_t^2 = \frac{\rho(\mu(t, Y_t) - r)}{\sigma(t, Y_t)}. \tag{34}$$

If $(\lambda_t^1, \lambda_t^2)$ further satisfies the Novikov condition, then it follows from Girsanov's theorem that Q is equivalent to P with density given by

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \lambda_t dW_t - \frac{1}{2}\int_0^T \|\lambda_t\|^2 dt\right). \tag{35}$$

5. Reciprocal Affine Models

Take μ and r to be constants and consider the case where $\sigma(t, Y_t) = \sqrt{Y_t}$, so that (32) becomes

$$R_t = R(t, Y_t) = \frac{(1-\rho^2)(\mu - r)^2}{2Y_t}. \tag{36}$$

Affine models form a well-studied class of interest rate models, often leading to analytic expressions for quantities such as bond prices. We can carry the results from these models to our problem by hypothesizing that R_t follows an affine process. We illustrate the idea in the specific case of the CIR model

$$dR_t = \alpha(\bar{R} - R_t)dt + \beta\sqrt{R_t}\left[\rho d\widetilde{W}_t^1 + \sqrt{1-\rho^2}d\widetilde{W}_t^2\right], \tag{37}$$

for constants $\alpha, \beta, \bar{R} > 0$.

Pricing and Hedging Formulas

The indifference price and Davis price of a volatility claim under the reciprocal affine models of the previous section both require computation of expressions of the form

$$I := E_t^Q \left[e^{-\int_t^T R_s ds} g(R_T) \right] \tag{38}$$

for functions $g : \mathbb{R}^+ \to \mathbb{R}$. Provided g is truncated in a careful way we have

$$g(R) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuR} \widehat{g}(u) du, \tag{39}$$

where

$$\widehat{g}(u) = \int_{-\infty}^{\infty} e^{iuR} g(R) dR. \tag{40}$$

Using Fubini's theorem, we have

$$I(R_t, t, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(t, T; u) \widehat{g}(u) du \tag{41}$$

where Ψ can be computed following Duffie et al (2000):

$$\begin{aligned} \Psi(u, R_t, t, T) &:= E_t^Q \left[e^{-\int_t^T R_s ds} e^{-iuR_T} \right] \\ &= \exp[A(t, T, u) + B(t, T, u) R_t]. \end{aligned} \tag{42}$$

Here

$$\begin{aligned} B(t, T, u) &= \frac{(b_2 + iu) b_1 - (b_1 + iu) b_2 e^{\Delta (t-T)}}{(b_2 + iu) - (b_1 + iu) e^{\Delta (t-T)}} \\ A(t, T, u) &= \frac{-2\alpha \bar{R}}{\beta^2} \log \left(\frac{b_2 + iu}{b_2 - B} \right) + \alpha \bar{R} b_1 (t - T) \end{aligned} \tag{43}$$

with $b_2 > b_1$ being the two roots of $x^2 - \frac{2\alpha}{\beta^2} x - \frac{2}{\beta^2}$ and $\Delta = \sqrt{\alpha^2 + 2\beta^2}$.

Setting $g(R_T) = e^{\gamma(1-\rho^2)B(R_T)}$, we obtain that the indifference price of the volatility claim $B = B(R_T)$ is simply

$$\pi^B = \frac{1}{\gamma(1-\rho^2)} \log \left[\frac{I(R_t, t, T)}{\Psi(0, R_t, t, T)} \right]. \tag{44}$$

Finally, the number of shares of stock to be held in order to optimally hedge against the claim B is

$$h^B(t, y) = \frac{1}{s} \left[\frac{b\rho}{\gamma(1-\rho^2)\sqrt{y}} \frac{\partial_y I}{I} + \frac{(\mu - r)}{\gamma y} \right], \tag{45}$$

whereas the number of shares held in the Merton portfolio is

$$h^0(t, y) = \frac{1}{s} \left[\frac{b\rho}{\gamma(1-\rho^2)\sqrt{y}} \frac{\partial_y \Psi(0)}{\Psi(0)} + \frac{(\mu - r)}{\gamma y} \right]. \tag{46}$$