

Games and Options in Incomplete Markets

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Strategic Decision Making

Suppose we want to assign **monetary values** to the strategic decision to:

- ▶ create a new firm;
- ▶ invest in a new project;
- ▶ start a real estate development;
- ▶ finance R&D;
- ▶ abandon a non-profitable project;
- ▶ temporarily suspend operations under adverse conditions and reactive them when conditions improve.

Valuation Elements

In all of the previous problems, we can identify the following common elements:

- ▶ uncertainty about the future;
- ▶ some degree of irreversibility;
- ▶ timing and managerial flexibility;
- ▶ interaction with other people's decisions.

To account for these elements, we are going to base our decisions on **values** obtained using the following theoretical tools:

- ▶ Net Present Value
- ▶ Real Options
- ▶ Game Theory

Net Present Value

- ▶ Net Present Value takes into account the **intrinsic** advantages of a given investment when compared to capital markets.
- ▶ This are essentially due to market imperfections, such as entry barriers, product differentiation, economy of scale, etc...
- ▶ Denoting the expected present value of *future cash flows* of a given project by \tilde{V} and the corresponding *sunk cost* by I , then its NPV is

$$\text{NPV} = \tilde{V} - I$$

- ▶ The decision rule according to NPV is to invest whenever $\tilde{V} > I$.

The Real Options Approach

- ▶ If we view the project as an underlying asset, then an investment opportunity with a sunk cost I is the *formal* analogue of an American call option on \tilde{V} with strike price I .
- ▶ The Real Options Approach applies techniques used for **financial** options to determine the value \tilde{C} for the option to invest.
- ▶ Therefore, an investor possessing an opportunity with value \tilde{C} will invest only when

$$\tilde{V} - I > \tilde{C}.$$

- ▶ This results in **higher** exercise thresholds, taking into account the **value of waiting**.

Successes and Limitations

- ▶ According to a recent survey, 26% of CFOs in North America “always or almost always” consider the value of real options in projects.
- ▶ This is due to familiarity with the option valuation paradigm in financial markets and its lessons.
- ▶ **But** most of the literature in Real Options is based on one or both of the following assumptions: (1) **infinite time horizon** and (2) perfectly correlated **spanning asset** .
- ▶ Though some problems have long time horizons (30 years or more), most strategic decisions involve much shorter times.
- ▶ The vast majority of underlying projects are **not** perfectly correlated to any asset traded in financial markets.

Alternatives

- ▶ The use of well-known numerical methods (e.g. finite differences) allows for finite time horizons.
- ▶ As for the spanning asset assumption, the absence of perfect correlation with a financial asset leads to an **incomplete market**.
- ▶ Replication arguments can no longer be applied to value managerial opportunities.
- ▶ Instead, one needs to rely on **risk preferences**.
- ▶ The most widespread way to do this in the strategic decision making literature is to introduce an **internal rate of return**, which replaces the risk-free rate, and use dynamic programming.
- ▶ This approach lacks the intuitive understanding of opportunities as **options**.
- ▶ We prefer to stick with the options paradigm and use utility-based methods to calculate their values.

A one-period investment model

- ▶ Consider a two-factor market where the **discounted** prices for the project V and a correlated traded asset S follow:

$$(S_T, V_T) = \begin{cases} (uS_0, hV_0) & \text{with probability } p_1, \\ (uS_0, \ell V_0) & \text{with probability } p_2, \\ (dS_0, hV_0) & \text{with probability } p_3, \\ (dS_0, \ell V_0) & \text{with probability } p_4, \end{cases} \quad (1)$$

where $0 < d < 1 < u$ and $0 < \ell < 1 < h$, for positive initial values S_0, V_0 and historical probabilities p_1, p_2, p_3, p_4 .

- ▶ Let the risk preferences be specified through an exponential utility $U(x) = -e^{-\gamma x}$.
- ▶ An investment opportunity is model as an option with **discounted** payoff $C_t = (V - e^{-rt}I)^+$, for $t = 0, T$.

European Indifference Price

- ▶ Without the opportunity to invest in the project V , a rational agent with initial wealth x will try to solve the optimization problem

$$u^0(x) = \max_H E[U(X_T^x)], \quad (2)$$

where

$$X_T^x = \xi + HS_T = x + H(S_T - S_0). \quad (3)$$

is the wealth obtained by keeping ξ dollars in a risk-free cash account and holding H units of the traded asset S .

- ▶ An agent with initial wealth x who pays a price π for the opportunity to invest in the project will try to solve the modified optimization problem

$$u^C(x - \pi) = \max_H E[U(X_T^{x-\pi} + C_T)] \quad (4)$$

- ▶ The **indifference price** for the option to invest in the final period as the amount π^C that solves the equation

$$u^0(x) = u^C(x - \pi). \quad (5)$$

Explicit solution

Denoting the two possible pay-offs at the terminal time by C_h and C_ℓ , the European indifference price defined in (5) is given by

$$\pi^C = g(C_h, C_\ell) \quad (6)$$

where, for fixed parameters $(u, d, p_1, p_2, p_3, p_4)$ the function $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$g(x_1, x_2) = \frac{q}{\gamma} \log \left(\frac{p_1 + p_2}{p_1 e^{-\gamma x_1} + p_2 e^{-\gamma x_2}} \right) + \frac{1-q}{\gamma} \log \left(\frac{p_3 + p_4}{p_3 e^{-\gamma x_1} + p_4 e^{-\gamma x_2}} \right), \quad (7)$$

with

$$q = \frac{1-d}{u-d}.$$

Early exercise

- ▶ When investment at time $t = 0$ is allowed, it is clear that immediate exercise of this option will occur whenever its **exercise value** $(V_0 - I)^+$ is larger than its **continuation value** π^C .
- ▶ That is, from the point of view of this agent, the value at time zero for the opportunity to invest in the project either at $t = 0$ or $t = T$ is given by

$$C_0 = \max\{(V_0 - I)^+, g((hV_0 - e^{-rT}I)^+, (\ell V_0 - e^{-rT}I)^+)\}.$$

A multi-period model

- ▶ Consider now a continuous-time two-factor market of the form

$$dS_t = (\mu_1 - r)S_t dt + \sigma_1 S_t dW$$

$$dV_t = (\mu_2 - r)V_t dt + \sigma_2 V_t (\rho dW + \sqrt{1 - \rho^2} dZ).$$

- ▶ We want to approximate this market by a discrete-time processes (S_n, V_n) following the one-period dynamics (1).
- ▶ This leads to the following choice of parameters:

$$u = e^{\sigma_1 \sqrt{\Delta t}}, \quad h = e^{\sigma_2 \sqrt{\Delta t}},$$

$$d = e^{-\sigma_1 \sqrt{\Delta t}}, \quad \ell = e^{-\sigma_2 \sqrt{\Delta t}},$$

$$p_1 + p_2 = \frac{e^{(\mu_1 - r)\Delta t} - d}{u - d}, \quad p_1 + p_3 = \frac{e^{(\mu_2 - r)\Delta t} - \ell}{h - \ell}$$

$$\rho \sigma_1 \sigma_2 \Delta t = (u - d)(h - \ell)[p_1 p_4 - p_2 p_3],$$

supplemented by the condition $p_1 + p_2 + p_3 + p_4 = 1$.

Grid Values

- ▶ Instead a triangular tree for project values, we consider a $(2M + 1) \times N$ rectangular grid whose repeated columns are given by

$$V^{(i)} = h^{M+1-i} V_0, \quad i = 1, \dots, 2M + 1, \quad (8)$$

ranging from $(h^M V_0)$ to $(\ell^M V_0)$.

- ▶ Then each realization for the discrete-time process V_n following the dynamics (1) can then be thought of as a path over this grid.

Option pricing on the grid

- ▶ We determine the **discounted** value of the option to invest on the project as a function C_{in} on this grid.
- ▶ We start by with the boundary conditions:

$$\begin{aligned}C_{iN} &= (V^{(i)} - e^{-rT}I)^+, & i &= 1, \dots, 2M + 1, \\C_{1n} &= V^{(1)} - e^{-rn\Delta t}, & n &= 0, \dots, N, \\C_{2M+1,n} &= 0, & n &= 0, \dots, N.\end{aligned}$$

- ▶ Values in the interior of the grid are then obtained by backward induction as follows:

$$C_{in} = \max \left\{ (V^{(i)} - e^{-rn\Delta t}I)^+, g(C_{i+1,n+1}, C_{i-1,n+1}) \right\}. \quad (9)$$

- ▶ For each time t_n , the *exercise threshold* V_n^* is defined as the project value for which the exercise value becomes higher than its continuation value.

Numerical Experiments - Act I

- ▶ We now investigate how the exercise threshold varies with the different model parameters.
- ▶ The fixed parameters are

$$\begin{aligned}I &= 1, & r &= 0.04, & T &= 10 \\ \mu_1 &= 0.115, & \sigma_1 &= 0.25, & S_0 &= 1 \\ \sigma_2 &= 0.2, & V_0 &= 1\end{aligned}$$

- ▶ Given these parameters, the CAPM equilibrium expected rate of return on the project for a given correlation ρ is

$$\bar{\mu}_2 = r + \rho \left(\frac{\mu_1 - r}{\sigma_1} \right) \sigma_2. \quad (10)$$

- ▶ The difference $\delta = \bar{\mu}_2 - \mu_2$ is the **below-equilibrium rate-of-return shortfall** and plays the role of a dividend rate paid by the project, which we fix at $\delta = 0.04$.

Known Thresholds

- ▶ In the limit $\rho \rightarrow \pm 1$ (complete market), the closed-form expression for the investment threshold obtained in the case $T = \infty$ gives $V_{DP}^* = 2$.
- ▶ This should be contrasted with the NPV criterion (that is, invest whenever the net present value for the project is positive) which in this case gives $V_{NPV}^* = 1$.
- ▶ The limit $\gamma \rightarrow 0$ in our model corresponds to the McDonald and Siegel (1986) threshold, obtained by assuming that investors are averse to market risk but neutral towards idiosyncratic risk.
- ▶ For our parameters, the adjustment to market risks is accounted by CAPM and this threshold coincides with $V_{DP}^* = 2$

Dependence on Correlation and Risk Aversion

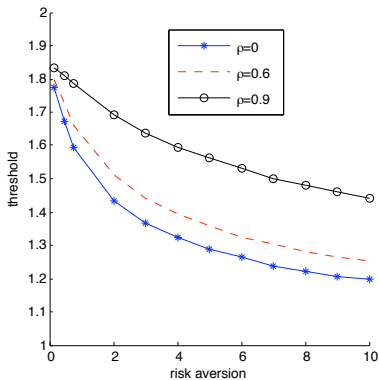
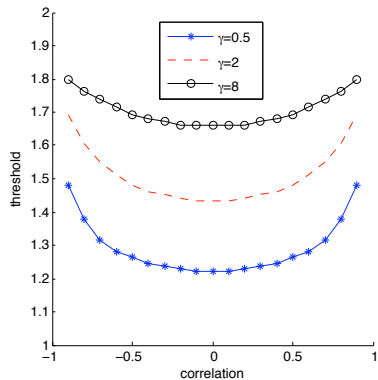


Figure: Exercise threshold as a function of correlation and risk aversion.

Dependence on Volatility and Dividend Rate

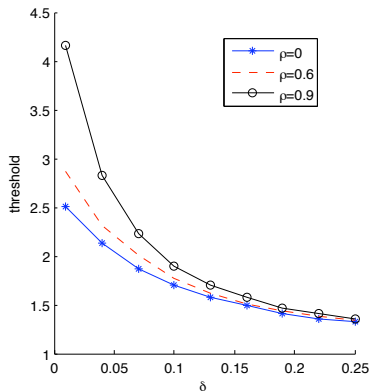
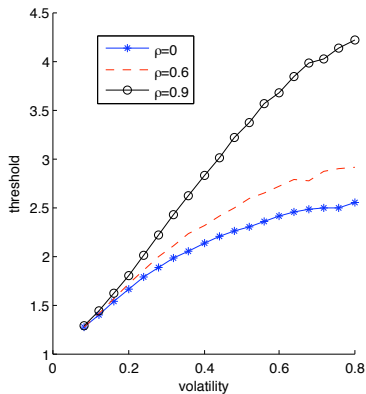


Figure: Exercise threshold as a function of volatility and dividend rate.

Dependence on Time to Maturity

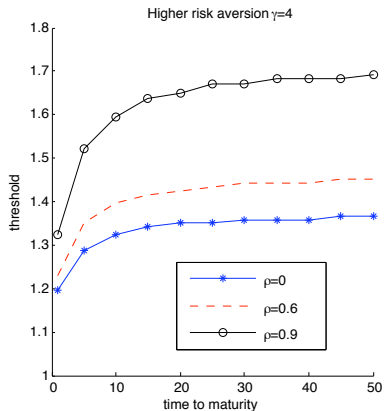
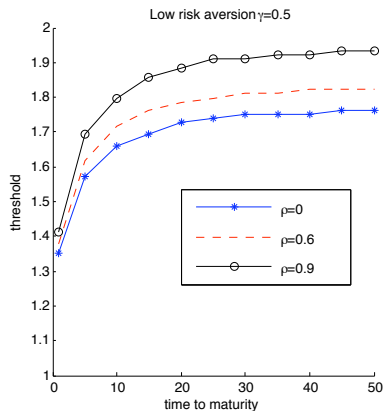


Figure: Exercise threshold as a function of time to maturity.

Values for the Option to Invest

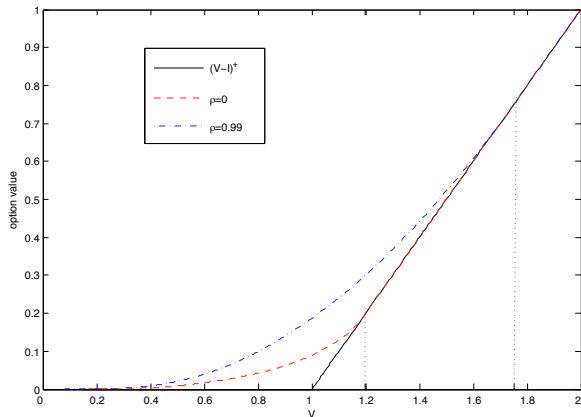


Figure: Option value as a function of underlying project value. The threshold for $\rho = 0$ is 1.1972 and the one for $\rho = 0.99$ is 1.7507.

Suspension, Reactivation and Scrapping

- ▶ The previous framework ignores the possibility of negative cash flows arising from the active project, for instance, when operating costs exceed the revenue.
- ▶ We have to consider the option to abandon the project when such cash flows become too negative.
- ▶ Instead of completely abandoning the project, we might have the option to “mothball” it by paying a sunk cost E_M and a maintenance rate $m < C$.
- ▶ Once prices for the output become favorable again, we have the option to reactive the project by paying a sunk cost $R < I$.
- ▶ Finally, if prices drop too much, we have the option to completely abandon the project by paying a sunk cost E_S (which could be negative, corresponding to a “scrap value”).
- ▶ As before, the decisions to invest, mothball, reactivate and scrap are triggered by the price thresholds $P_S < P_M < P_R < P_H$.

Project values and options

- ▶ Let us denote the value of an idle project by F^0 , an active project by F^1 and a mothballed project by F^M .
- ▶ Then

F^0 = option to invest at cost I

F^1 = cash flow + option to mothball at cost E_M

F^M = cash flow + option to reactivate at cost R
+ option to scrap at cost E_S

- ▶ We obtain its value on the grid using the recursion formula

$$F^k(i, j) = \max\{\text{continuation value, possible exercise values}\}.$$

Numerical Experiments - Act II

- ▶ We calculate these thresholds by keeping track of three simultaneous grids of option values.
- ▶ The fixed parameters now are

$$\mu_1 = 0.12, \quad \sigma_1 = 0.2, \quad S_0 = 1$$

$$\sigma_2 = 0.2, \quad V_0 = 1$$

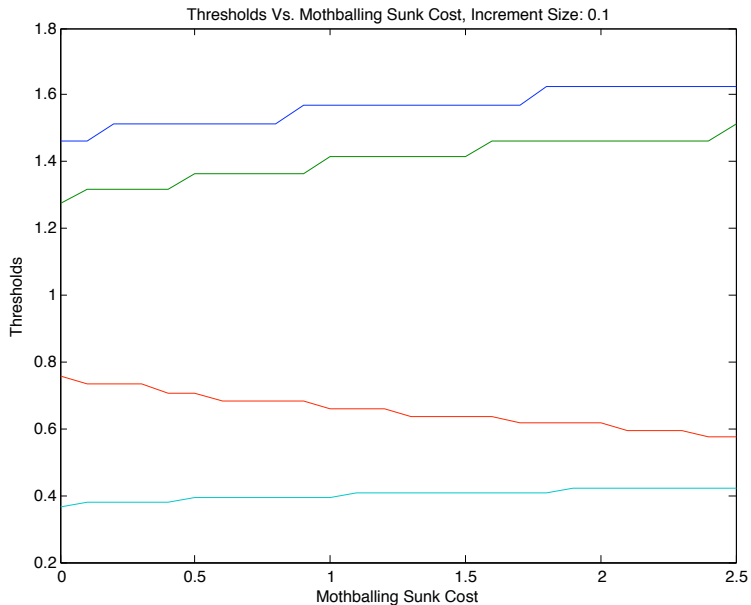
$$r = 0.05, \quad \delta = 0.05, \quad T = 30$$

$$l = 2, \quad R = 0.79, \quad E_M = E_S = 0$$

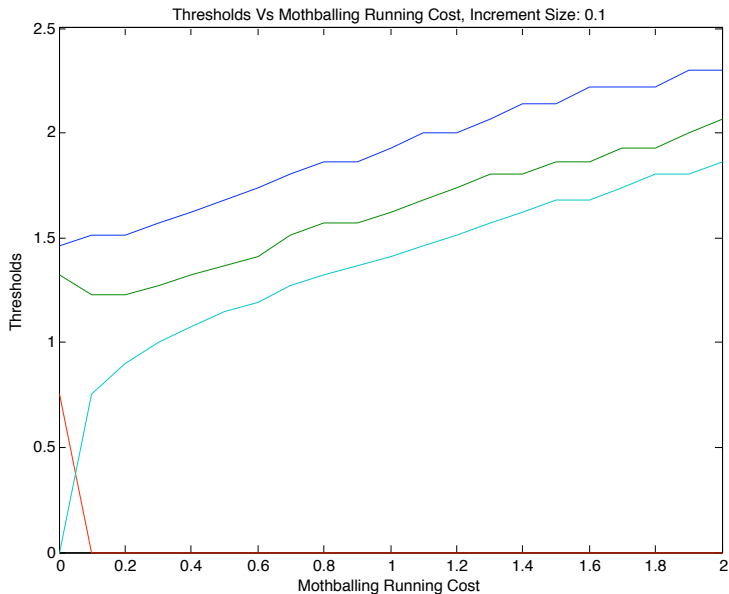
$$C = 1, \quad m = 0.01$$

$$\rho = 0.9, \quad \gamma = 0.1$$

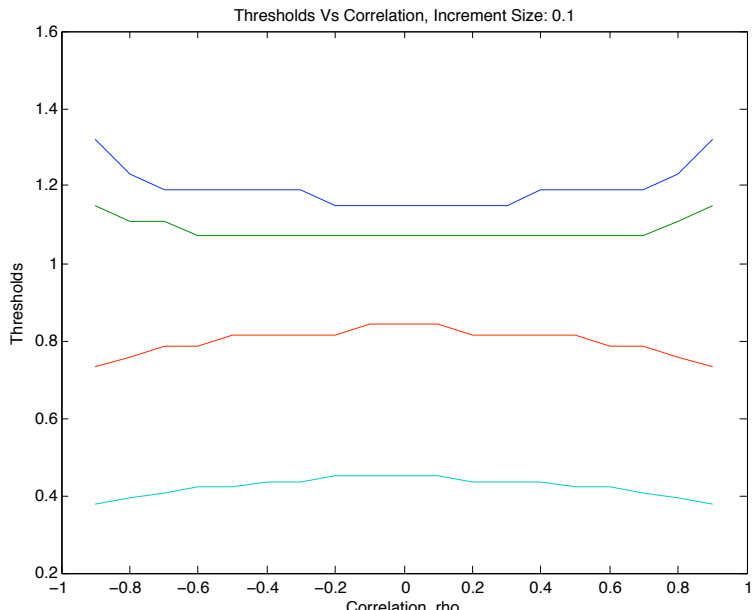
Dependence on Mothballing Sunk Cost



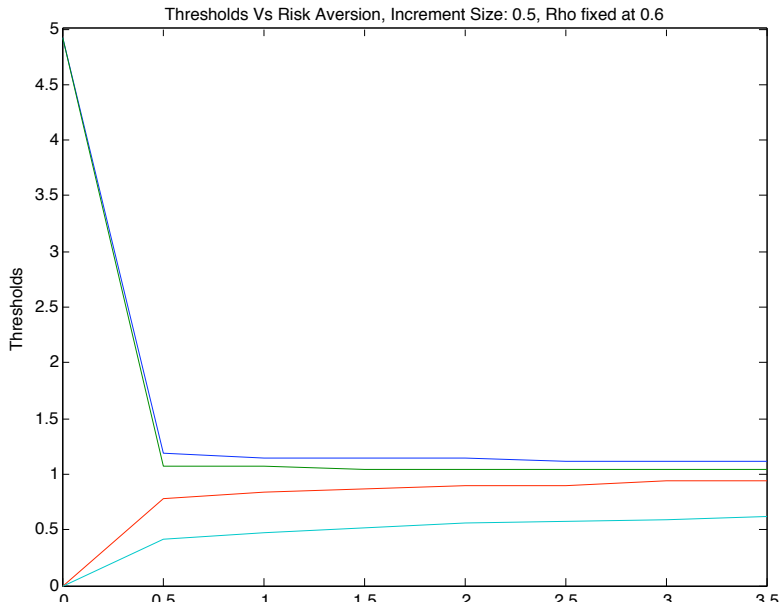
Dependence on Mothballing Running Cost



Dependence on Correlation



Dependence on Risk Aversion



Introducing Competition

- ▶ We use game theoretical tools to introduce the effect of competition.
- ▶ The goal is to assign a strategic value \tilde{G} to both conditional and unconditional moves toward investment that can create a competitive advantage in the market.
- ▶ This is then added to the NPV of a project.
- ▶ Therefore, the decision rule is to invest whenever

$$(\tilde{V} - I) + \tilde{G} > \tilde{C}.$$

Combining options and games

- ▶ For a systematic application of both **real options** and **game theory** in strategic decisions, we consider the following rules:
 1. Outcomes of a given game that involve a “wait-and-see” strategy should be calculated by option value arguments.
 2. Once the Nash equilibrium (NE) for a given game is found on a decision node, its value becomes the pay-off for an option at that node.
- ▶ In this way, option valuation and game theoretical equilibrium become **dynamically related** in a decision tree.
- ▶ In what follows, we denote the NE solution for a given game in bold face within the matrix of outcomes (rounded to nearest integer).

One Stage Strategic Investment

- ▶ As a first example, consider two **symmetric** firms contemplating a total investment $I = 80$ on a project with $V_0 = 100$ and equal probabilities to move up to $V^u = 200$ and down to $V^d = 50$.
- ▶ We take $u = 3/2$, $h = 2$, $p_1 = p_4 = 255/256$, $p_2 = p_3 = 1/256$, $\gamma = 0.1$, $r = 0$.
- ▶ Therefore, using expression (7) to calculate the *option value* for the “wait-and-see” strategy, we have the following matrix of outcomes for this game:

		B	
		Invest	Wait
A	Invest	(10,10)	(20,0)
	Wait	(0,20)	(11,11)

- ▶ For comparison, the complete market gives an option value of 48 to be shared by the firms.

Two-stage competitive R&D

- ▶ Suppose now that firm A is the only firm facing an R&D investment at cost $I_0 = 25$ at time t_0 , whereas at time t_1 the firms can equally share the follow-on cost $I_1 = 80$.
- ▶ We will assume that the technology resulting from the R&D investment is **proprietary**, so that the market share of firm A after the R&D phase is $s = 3/5$.
- ▶ Moreover, we assume that the market value continues to evolve from time t_1 to time t_2 following the same dynamics, that is, at time t_2 the possible market values in these two-period tree are

$$V^{uu} = 400, \quad V^{ud} = 100, \quad V^{dd} = 25.$$

Analyzing the game

- ▶ If demand is high at time t_1 ($V^u = 200$), we have:

		B (follower)	
		Invest	Wait
A (leader)	Invest	(80,40)	(120,0)
	Wait	(0,120)	(42,22)

- ▶ If demands is low at time t_1 ($V^d = 60$), we have:

		B (follower)	
		Invest	Wait
A (leader)	Invest	(-10,-20)	(-30,0)
	Wait	(0,-30)	(8,0)

- ▶ Then $C_A = -I_0 + g(80, 8) = -25 + 30 = 5 > 0$,
- ▶ whereas $C_B = g(40, 0) = 15$
- ▶ Therefore the R&D investment is recommended for A.
- ▶ For comparison, the complete market results are $C_A = 10$ and $C_B = 7$.