RECONSTRUCTION RESULTS FOR FIRST-ORDER THEORIES

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By:

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Abstract

In this thesis, we study problems related to the reconstruction (up to bi-interpretability) of first-order theories from various functorial invariants: automorphism groups, endomorphism monoids, (categories of) countable models, and (ultra)categories of models.

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Chapter 1

Introduction

Let T be a first-order theory. Any formula $\varphi(x)$ of T^{eq} (so a definable set of T quotiented by a definable equivalence relation of T) induces a "functor of points" $\mathrm{ev}_{\varphi(x)}$ on the category $\mathbf{Mod}(T)$ of models of T with maps the elementary embeddings, by sending $M \mapsto \varphi(M)$. In this way the category $\mathbf{Def}(T)$ of 0-definable sets of T embeds into the category of functors $[\mathbf{Mod}(T), \mathbf{Set}]$, via the "evaluation map" $\mathrm{ev} : T \to [\mathbf{Mod}(T), \mathbf{Set}]$.

Here is the motivating problem: how do we recognize, up to isomorphism, the image of ev inside $[\mathbf{Mod}(T), \mathbf{Set}]$? This would give a way of reconstructing the theory T from its category of models $\mathbf{Mod}(T)$. That is, given an arbitrary functor $X : \mathbf{Mod}(T) \to \mathbf{Set}$—some way of attaching a set to every model of T, functorial with respect to elementary embeddings—how can we tell if X was isomorphic to some functor of points $\mathrm{ev}_{\varphi(x)}$ for some formula $\varphi(x) \in T^{\mathrm{eq}}$? We call such functors X *definable.*

A necessary condition for definability is compatibility with ultraproducts. Łos' theorem 3.2.1 tells us that evaluation functors $\mathrm{ev}_{\varphi(x)}$ commute with ultraproducts, that is,

$$\varphi\left(\prod_{i \to \mathcal{U}} M_i\right) = \prod_{i \to \mathcal{U}} \varphi(M_i).$$

Strong conceptual completeness for first-order logic, as proved by Makkai in [12], provides a sort of converse to Łos' theorem, and says that the definable functors are precisely the ones which preserve ultraproducts and certain formal comparison maps between ultraproducts, called *ultramorphisms*, which generalize the diagonal embeddings of models into their ultrapowers. This recovers T up to bi-interpretability. To precisely state Makkai's result, we must formalize what it means for an arbitrary functor $X : \mathbf{Mod}(T) \to \mathbf{Set}$ to "preserve ultraproducts" and "preserve" these ultramorphisms. This motivates the formalism of ultracategories, which we review in chapter 3.

Any general framework which recovers theories from their categories of models should be considerably simplified for $\aleph_0$-categorical theories, whose definable sets are exceptionally easy to understand (being precisely the finite disjoint unions of orbits of the automorphism group) and in fact are determined up to bi-interpretability by the automorphism group of the unique countable model topologized by pointwise convergence.

We will show (Theorem 4.3.2) that when T is $\aleph_0$-categorical, we can check definability by checking compatibility with ultraproducts and just diagonal embeddings into ultrapowers, so that for $\aleph_0$-categorical theories, the definability criteria provided by strong conceptual completeness can indeed be simplified.

By modifying our techniques, we will deduce the full statement of strong conceptual completeness for $\aleph_0$-categorical T (Theorem 7.2.2) from just the preservation of diagonal embeddings into ultrapowers. This will follow as a corollary of a general definability criterion (Theorem 7.2.1) for recognizing the evaluation functors of definable sets among the evaluation functors for objects in the classifying topos of any first-order theory T.

Finally, in chapter 8, we construct counterexamples to Theorem 7.2.2 when the assumption of $\aleph_0$-categoricity is removed.

Chapter 2

Basic model theory and categorical logic

2.1 Introduction

In this chapter, we develop the necessary categorical logic (and some model-theoretic consequences) for our main results. We assume familiarity with the basics of first-order logic and model theory, e.g. the first few chapters of [14]. We also assume familiarity with basic category theory, e.g. the first few chapters of [9].

2.1.1 Notation and conventions

- Unless explicitly stated otherwise, we are always working in multisorted classical first-order logic.
- Unadorned variables in formulas will generally stand for finite tuples of appropriately-sorted variables.
- Similarly, when we say "sort" we mean a finite tuple of sorts. When we wish to

stress that a sort is not a finite tuple of other sorts, we will say "basic sort".

- If we have already mentioned a tuple of variables x, then we will write S_x for the sort corresponding to x.
- φ, ϕ, ψ, and θ will usually mean first-order formulas.
- If $\mathcal{L}$ is a first-order language, we write $\mathsf{Functions}(\mathcal{L})$, $\mathsf{Relations}(\mathcal{L})$, $\mathsf{Constants}(\mathcal{L})$, and $\mathsf{Formulas}(\mathcal{L})$ to mean the collections of function symbols, relation symbols, constant symbols, and first-order $\mathcal{L}$-formulas, respetively.
- If X is a set, we write 2^X for the power set $\{S \,|\, S \subseteq X\}$.

2.2 Basic notions

2.2.1 The category of definable sets

The starting point for first-order categorical logic is the identification of a theory with its category of definable sets.

Definition 2.2.1. Let T be a first-order $\mathcal{L}$-theory. The **category of definable sets** comprises:

$$\begin{cases} \text{Objects: } \mathsf{Formulas}(\mathcal{L}) /_{\sim}, \text{ where } \phi(x) \sim \psi(x) \iff \phi(M) = \psi(M) \text{ for all } M \models T, \\ \text{Morphisms: } \mathbf{Def}(T)(\varphi(x), \psi(y)) \overset{\text{df}}{=} \left(\{\phi \in \mathsf{Formulas}(\mathcal{L}) \,|\, T \models \phi \text{ is a function } \varphi(x) \to \psi(y)\} /_{\sim} \right) \end{cases}$$

Some remarks:

1. In the above, we are defining morphisms to be equivalence classes of graphs of definable functions, where we are using the same equivalence relation as we did for objects.
2. Everything so far is 0-*definable*, and will remain so unless stated otherwise.

3. By the completeness theorem for first-order logic, the notion of equivalence of formulas used in defining the objects of $\mathbf{Def}(T)$ is the same as T*-provable equivalence*: $\varphi(x) \sim \psi(y) \iff T \vdash \varphi(x) \leftrightarrow \psi(x)$. By the downward Lowenheim-Skolem theorem, it also suffices to check $\sim$-equivalence by seeing if two formulas have the same points on models whose sizes are less than or equal to the size of the theory.

4. T always has an empty product of sorts, which we think of as a generic singleton set 1. If T interprets a constant in a sort S, then we think of it as a nullary function $1 \to S$ in $\mathbf{Def}(T)$.

Below, we collect some observations on how certain categorical operations and category-theoretic properties of $\mathbf{Def}(T)$ correspond to first-order logic in models of T.

Remark 2.2.2. To know that a formula $\varphi(x)$ lives in a sort B is to specify an embedding of the definable set $\varphi(x) \hookrightarrow B$. If $\varphi(x)$ and $\psi(x)$ are two definable sets in T both of the same sort B, then $\varphi(x) \wedge \psi(x)$ is the pullback

$$\begin{tikzcd}
\varphi(x) \arrow[r] & B \\
\varphi(x) \wedge \psi(x) \arrow[u] \arrow[r] & \psi(x). \arrow[u]
\end{tikzcd}$$

Remark 2.2.3. Dually, $\varphi(x) \vee \psi(x)$ is the pushout of $\varphi(x)$ and $\psi(x)$ over $\varphi(x) \wedge \psi(x)$.

Remark 2.2.4. $\mathbf{Def}(T)$ has an initial object $0 = \varnothing$. It is also **strict**: any map into 0 is an isomorphism.

Remark 2.2.5. The existence of complements means that for every subobject $\varphi(x) \hookrightarrow B$, there exists a unique (up to isomorphism) subobject $\neg\varphi(x) \hookrightarrow B$ such that:

1. The meet $\varphi(x) \wedge \neg\varphi(x)$ is 0.

2. The join $\varphi(x) \vee \neg\varphi(x)$ is B.

An immediate consequence of our definitions (and a basic sanity check) is that the operations of first-order logic inside $\mathbf{Def}(T)$ may be checked inside any model:

Definition 2.2.6. Let $M \models T$. Then M is the data of a functor $M : \mathbf{Def}(T) \to \mathbf{Set}$ ("taking M-points"; "passage to a model"; etc.) Explicitly, it is given by

$$\big(\varphi(x) \xrightarrow{f} \psi(y)\big) \mapsto \big(\varphi(M) \xrightarrow{f(M)} \psi(M)\big),$$

where $\varphi(M)$, $f(M)$, and $\psi(M)$ are the interpretations of φ, f, and ψ in the model M.

We write $\mathbf{Def}_M(T)$ to denote the image of this functor ("the category of 0-definable sets in M".)

Lemma 2.2.7. *The inclusion $\mathbf{Def}_M(T)$ preserves and reflects finite limits (in fact creates them.)*

Proof. By the canonical product-equalizer decomposition (see [9], V.2.2.) for limits, it suffices to check the preservation and reflection of limits on just products and equalizers.

The usual construction of an equalizer of two maps $f, g : X \to Y$ in $\mathbf{Set}$ is always definable: it is the subset of X consisting of those elements x such that $f(x) = g(x)$.

Similarly, if X and Y are definable, then $X \times Y$ is definable, and the projections $X \times Y \overset{\pi_X}{\underset{\rightrightarrows}{\pi_Y}} X, Y$ are definable.

If J is a finite diagram in $\mathbf{Def}_M(T)$ and $\varprojlim J$ its limit, and $Z \in \mathbf{Def}_M(T)$ is a definable set in M equipped with a cone of definable maps to J, then Z has (in $\mathbf{Set}$) a unique mediating map to $\varprojlim J$, which is definable because it is definable in the cases when J is a product or equalizer diagram, the limit is finite, and by the canonical product-equalizer decomposition the mediating map for a general finite J is a composition of finitely many mediating maps for products and equalizers. □

2.2.2 Logical categories and elementary functors

One can try to isolate the categorical properties shared by those categories of the form $\mathbf{Def}(T)$ for T some first-order theory. This was done in Makkai-Reyes [13] and the resulting notion is that of a (Boolean) *logical category.*

Definition 2.2.8. A category $\mathbf{C}$ is a **logical category** if it has all finite limits (equivalently, all binary products and equalizers), and furthermore:

1. $\mathbf{C}$ has images: if $f : X \to Y$ is a map in $\mathbf{C}$, then there is a subobject $\mathrm{im}(f)$ of Y such that f factors through $\mathrm{im}(f) \hookrightarrow Y$ which satisfies the following universal property: whenever there is a commutative triangle

$$\begin{tikzcd} X' \arrow[d] \arrow[dr, "g"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

then g factors uniquely through $\mathrm{im}(f)$.

2. $\mathbf{C}$ has finite sups of subobjects: given any finite collection of subobjects $S_1, \ldots, S_n$ of B, there exists a smallest subobject in the subobject poset of B among those subobjects greater than all the S_i.

3. Images and sups of subobjects in $\mathbf{C}$ are stable under pullback ("images and unions commute with taking preimages"):

   We require that the image of a map $f : X \to Y$ satisfy the following property: if $g : Z \to Y$ is another map, then in the following situation with the pullback square

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X \times_Y Z \arrow[u] \arrow[r, "\pi_Z"'] & Z, \arrow[u, "g"'] \end{tikzcd}$$

   the pullback of $\mathrm{im}(f) \hookrightarrow Y$ along g is the same thing as $\mathrm{im}(\pi_Z)$.

   We require that for any finite collection of subobjects $S_1, \ldots, S_n$ of B with sup $\bigvee_i S_i \hookrightarrow B$ and any map $g : Z \to B$, then in the following situation with the pullback square

$$\begin{tikzcd} \bigvee_i S_i \arrow[r] & B \\ \bigvee_i S_i \times_B Z \arrow[u] \arrow[r, "\pi_Z"'] & Z, \arrow[u, "g"'] \end{tikzcd}$$

the subobject $\bigvee_i S_i \times_B Z$ of Z is the same thing as $\bigvee_i \{S_1 \times_B Z, \ldots, S_n \times_B Z\}$.

Furthermore $\mathbf{C}$ is called **Boolean** if every subobject has a complement, in the sense of 2.2.5.

There is an obvious notion of maps between logical categories. In [13] these are called, aptly, *logical functors*, but after introducing pretoposes (Definition 2.6.16) we will work with pretoposes almost exclusively, and so we follow the terminology of [12], wherein logical functors between pretoposes are called *elementary*.

Definition 2.2.9. Let $\mathbf{C}$ and $\mathbf{C}'$ be logical categories. An **elementary functor** $\mathbf{C} \to \mathbf{C}'$ is a functor which preserves finite limits, finite sups of subobjects, and images.

Before we proceed, we verify, as claimed, that $\mathbf{Def}(T)$ is always a Boolean logical category.

Proposition 2.2.10. *Let T be a first-order theory. Then $\mathbf{Def}(T)$ is a Boolean logical category.*

Proof. 1. $\mathbf{Def}(T)$ has all binary products and equalizers: if $\varphi(x)$ and $\psi(y)$ are formulas, then we form their product $\varphi(x) \times \psi(y)$ as follows: replacing x and y with identically-sorted variables as necessary so that x and y are disjoint, we put $\varphi(x) \times \psi(y) \overset{\text{df}}{=} \varphi(x) \wedge \psi(y) \subseteq S_{xy}$.

Similarly, if we have a pair of definable functions, $\varphi(x) \underset{g}{\overset{f}{\rightrightarrows}} \psi(y)$, their equalizer is given by the formula $\varphi(x_1) \wedge \varphi(x_2) \wedge f(x_1) = g(x_2)$ (with x_1 and x_2 distinct variables.)

2. $\mathbf{Def}(T)$ has images: given a definable function f with graph relation $\Gamma(f)(x, y)$, the image $\mathrm{im}(f)$ of f is just the definable set $\exists x \Gamma(f)(x, y)$.

3. $\mathbf{Def}(T)$ has finite sups: given any finite collection $\varphi_1(x), \ldots, \varphi_n(x)$ of formulas such that for all $1 \leqslant i \leqslant n$, $T \models \forall x \varphi_i(x) \to \psi(x)$ (so the $\varphi_i(x)$ are subobjects of $\psi(x)$ in $\mathbf{Def}(T)$), their sup is just their join $\bigvee_{1 \leqslant i \leqslant n} \varphi_i(x) \to \psi(x)$.

   One checks that the monomorphisms in $\mathbf{Def}(T)$ are definable injections and that the pullback of two definable functions $\varphi_1(x) \xrightarrow{f} \psi(x) \xleftarrow{g} \varphi_2(x)$ is the subobject of the product $\varphi_1(x) \times \varphi_2(x)$ consisting of those pairs equalized by f and g. In particular, the pullback of a subobject along f is the preimage of the subobject along the definable function f. This implies that finite sups and images are pullback-stable.

$\square$

In the next section, we will review the non-categorical notions of interpretation between theories and structures in model theory, and show the extent to which these notions of interpretation are captured by logical categories and elementary functors between them.

We will then introduce the $(-)^{\mathrm{eq}}$-construction and a special class of logical categories called *pretoposes*, and show that pretoposes and elementary functors completely capture the notions of theories and interpretations.

2.3 Interpretations between theories and interpretations between structures

In this section, we review the notions of interpretations (abstractly between theories, and concretely between models) from model theory. We then show how these two notions are related. We then show that models of T are the same thing as elementary functors $\mathbf{Def}(T) \to \mathbf{Set}$, and prove that strict interpretations $T \to T'$ are the same thing as elementary functors $\mathbf{Def}(T) \to \mathbf{Def}(T')$.

2.3.1 Concrete interpretations

We will only define and work with concrete interpretations for one-sorted structures (although with a little care to make sure arities are preserved, the notion can be generalized to multi-sorted structures, by having functions $f : U_S \to M(S)$ for each sort S.)

Definition 2.3.1. Let M_1 be an $\mathcal{L}_1$-structure and let M_2 be an $\mathcal{L}_2$-structure. An **interpretation** $(f, f^*) : M_1 \to M_2$ is a surjection $f : U \twoheadrightarrow M_1$ where $U \subseteq M_2^k$, some $k \in \mathbb{N}$, such that the pullback $f^* : 2^{M_1} \to 2^{M_2}$ sends $\mathcal{L}_1$-definable sets of M_1 to $\mathcal{L}_2$-definable sets of M_2.

We call such an interpretation a **concrete interpretation**.

Definition 2.3.2. If, in the above definition, the function $f : U \twoheadrightarrow M_1$ is also injective, we say that (f, f^*) is a **strict concrete interpretation**.

Definition 2.3.3. (c.f. [1]) Let $(f_1, f_1^*), (g_1, g_1^*) : M \rightrightarrows M'$ be interpretations. We say that (f_1, f_1^*) is **homotopic** to (f_2, f_2^*), written $(f_1, f_1^*) \sim (g_1, g_1^*)$, if, writing U for the domain of f_1 and V for the domain of f_2, the equalizer relation

$$\mathrm{eq}(f_1, f_2) = \{(u, v) \,|\, u \in U, v \in V, f_1(u) = f_2(v)\}$$

is definable.

Definition 2.3.4. We additionally say that a homotopy is a **strict** homotopy if the equalizer relation in the above definition is the graph of a definable bijection. Two concrete interpretation are strict homotopic if and only if both concrete interpretations are strict.

Definition 2.3.5. (c.f. [1]) Let $(f, f^*) : M \to M'$ and let $(g, g^*) : M' \to M''$ be interpretations.

The **composite interpretation** $(g, g^*) \circ (f, f^*) = (g * f, (g * f)^*)$ is defined as follows: $g * f$ has domain $g^* U_f$ where U_f is the domain of f, and is given by the composition $\widehat{g} \circ f$ where $\widehat{g}$ is the canonical extension of g to $g^* U_f$.

Definition 2.3.6. (c.f. [1]) A concrete **bi-interpretation** between two structures M and M' is a pair of interpretations

$$(f, f^*) : M \leftrightarrows M' : (g, g^*)$$

such that $(gf, g^* f^*) \sim 1_M$ and $(fg, f^* g^*) \sim 1_{M'}$.

On the other hand, one can also define interpretations purely syntactically, between theories.

2.3.2 Abstract interpretations

The following definition seems to be folklore.

Definition 2.3.7. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages, so each equipped with a set of sorts, function, relation, and constant symbols with arities taken from the set of sorts.

Let $\mathsf{Symb}(\mathcal{L})$ comprise all the nonlogical symbols of $\mathcal{L}$.

An **interpretation** of languages I of $\mathcal{L}_1$ in $\mathcal{L}_2$ is an assignment comprising:

$$\begin{cases} \text{A map } I_0 : \mathsf{Sorts}(\mathcal{L}_1) \to \mathsf{Formulas}(\mathcal{L}_2), \text{and} \\ \text{a map } I_1 : \mathsf{Symb}(\mathcal{L}) \to \mathsf{Formulas}(\mathcal{L}_2), \end{cases}$$

(where we view the equality symbol of each sort as a definable relation) such that the maps are compatible with arity, i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Sorts}(\mathcal{L}_1) \arrow[r, "I_0"] & \mathsf{Formulas}(\mathcal{L}_2) \arrow[dr, "\text{arity}"] & \\
& & \mathsf{Sorts}(\mathcal{L}_2) \\
\mathsf{Symb}(\mathcal{L}_1) \arrow[uu, "\text{arity}"] \arrow[r, "I_1"'] & \mathsf{Formulas}(\mathcal{L}_2) \arrow[ur, "\text{arity}"'] &
\end{tikzcd}$$

where we define the arity of a formula to be the sorts of its tuple of free variables.

Remark 2.3.8. Each interpretation of languages $I : \mathcal{L}_1 \to \mathcal{L}_2$ induces a map(by induction on complexity of formulas) $I : \mathsf{Formulas}(\mathcal{L}_1) \to \mathsf{Formulas}(\mathcal{L}_2)$, in particular a map $I : \mathsf{Sentences}(\mathcal{L}_1) \to \mathsf{Sentences}(\mathcal{L}_2)$.

So far, an interpretation of languages only requires that arities and sorts need to make sense. The following definition ensures that symbols are interpreted in a sensible way.

Definition 2.3.9. Let T_1 and T_2 be $\mathcal{L}_1$- and $\mathcal{L}_2$-theories. An interpretation of theories $I : T_1 \to T_2$ is an interpretation of languages $I : \mathcal{L}_1 \to \mathcal{L}_2$ such that

$$T_1 \models \psi \implies T_2 \models I(\psi).$$

Remark 2.3.10. If $T_1 \subseteq T_2$ is an inclusion of $\mathcal{L}$-theories, then the identity interpretation of $\mathcal{L}$ induces an interpretation of theories $T_1 \to T_2$.

It follows that when $T_1' \subseteq T_1$ is an inclusion of theories, any interpretation $T_1 \to T_2$ restricts to an interpretation $T_1' \to T_2$.

In particular, any interpretation of an $\mathcal{L}_1$-theory T_1 in an $\mathcal{L}_2$-theory T_2 extends an interpretation of the *empty* $\mathcal{L}_1$-theory in T_2. Since the empty theory in any language always proves that equality is an equivalence relation, equality must always be interpreted as an equivalence relation.

Example 2.3.11. Every theory has the identity interpretation with itself; more generally, every theory can be n-diagonally interpreted in itself: send each sort S to the diagonal of

$$S \times S \ldots (n \text{ times}) \cdots \times S$$

and induce the rest of the interpretation by restricting from sorts to the definable sets they contain. An explicit description in terms of a concrete interpretation of models is given at 2.3.21.

Definition 2.3.12. If an abstract interpretation interprets equality as equality, we say the interpretation is a **strict abstract interpretation** (in the literature, this is often called a **definition** of one theory inside another.)

In keeping with the traditional Ahlbrandt-Zeigler ([1]) treatment of bi-interpretations, which avoids imaginaries (for the definition of imaginaries and what it means to eliminate them, see 2.6.5), we define the abstract analogue of a concrete bi-interpretation (Definition 2.3.6).

Definition 2.3.13. An abstract **bi-interpretation** between two theories T and T' is a pair of abstract interpretations $F : T \to T'$ and $G : T' \to T$ such that:

For any definable set X of T there exists a definable surjection $\eta_X : GF(X) \twoheadrightarrow X$ whose kernel relation is equal to $GF(=)$, the definable equivalence relation interpreting equality (on the definable set X). Furthemore, the collection of η_X must satisfy the following naturality condition: for any definable function $X \xrightarrow{f} Y$ in T, the square

$$\begin{tikzcd} X \arrow[d, "f"'] & GF(X) \arrow[l, "\eta_X"'] \arrow[d, "GF(g)"] \\ Y & GF(Y) \arrow[l, "\eta_Y"] \end{tikzcd}$$

commutes, and dually

for any definable set X' of T' there exists a definable surjection $\epsilon_{X'} : FG(X') \twoheadrightarrow X'$ in T' whose kernel relation is equal to $FG(=)$, the definable equivalence relation interpreting equality on Y, such that for any definable function $X' \xrightarrow{f'} Y'$ in T', the square

$$\begin{tikzcd} FG(X') \arrow[r, "\epsilon_{X'}"] \arrow[d, "FG(f')"'] & X' \arrow[d, "f'"] \\ FG(Y') \arrow[r, "\epsilon_{Y'}"'] & Y' \end{tikzcd}$$

commutes.

We furthemore say that an abstract bi-interpretation is **strict** if all the maps η_X and ϵ_Y are bijective, not just surjective. An abstract bi-interpretation is strict if and only if its constituent abstract interpretations are strict.

2.3.3 Comparing abstract and concrete interpretations

Now we explicate the relationship between abstract and concrete interpretations.

Proposition 2.3.14. *(f, f^*) is a strict concrete interpretation $M_1 \to M_2$ if and only if f^* also restricts to an elementary functor $\mathbf{Def}_{M_1}(\mathbf{Th}(M_1)) \to \mathbf{Def}_{M_2}(\mathbf{Th}(M_2))$.*

Proof. We only have to show that an interpretation (f, f^*) always induces an elementary functor $\mathbf{Def}(\mathbf{Th}(M_1)) \to \mathbf{Def}(\mathbf{Th}(M_2))$.

Since the morphisms in these categories are already definable sets and the source and target maps the projections (which correspond to existential quantification), functoriality will follow from the preservation of the elementary operations.

Since f^* was induced by taking preimages along a function f, it preserves products (i.e. arity), conjunction, and negation.

Since f was surjective, f^* takes nonempty sets to nonempty sets, so existential statements continue to have witnesses, i.e. existential quantification.

Finally, if $R(\vec{c})$ is an atomic sentence in M_1, then $\vec{c} \in R^{M_1}$, and since f was a function, $f^*\vec{c} \in f^*R$, so f^* preserves atomic sentences.

Hence f^* induces an interpretation $\mathbf{Th}(M_1) \to \mathbf{Th}(M_2)$, and therefore must restrict to an elementary functor $\mathbf{Def}_{M_1}(\mathbf{Th}(M_1)) \to \mathbf{Def}_{M_2}(\mathbf{Th}(M_2))$. $\square$

Proposition 2.3.15. *Let $M \models T_1$ and $N \models T_2$ be structures, and let $(f, f^*) : M \to N$, $U \subseteq N^k$, $f : U \to M$ be a strict concrete interpretation of M in N. Then f^* induces an elementary functor $\mathbf{Def}(T_1) \to \mathbf{Def}(T_2)$.*

Proof. It suffices to see that f^* preserves $\wedge, \neg$ and $\exists$. The first two are preserved because f^* is given by taking preimages along a function. $\exists$ is preserved because f is assumed surjective: if $M \models \varphi(a, b)$, then $f^*\varphi$ is satisfied by the pair of imaginaries $f^*\{a\}$ and $f^*\{b\}$, so $f^*\{a\}$ satisfies $\exists x f^*\varphi(x, y)$ if and only if $f^*\{a\}$ satisfies $f^*(\exists x\varphi(x, y))$ if and only if $a \in \exists x\varphi(x, y)$. $\square$

Combining the previous proposition with Theorem 2.4.1, we get:

Corollary 2.3.16. *Strict concrete interpretations restrict to strict abstract interpretations.*

Remark 2.3.17. In the previous two propositions, strictness was necessary to even define a functor (resp. abstract interpretation) because we needed the preimage of a graph of a function to again be a graph of a definable function (resp. provably equivalent in the interpreting theory to the graph of a function), c.f. remark 2.4.2.

We now answer the question: given a concrete interpretation $(f, f^*) : M_1 \to M_2$, for $f : U \to M_1$, which other concrete interpretations (g, g^*) induce the same abstract interpretation as (f, f^*)? The next proposition says that any two concrete interpretations which restrict to the same abstract interpretation must be conjugate by an automorphism.

Proposition 2.3.18. *Let M, N, and U be as before. If (f, f^*) and (g, g^*) are both interpretations of M in N such that the domain of f and g are both U and f^* and g^* induce identical elementary functors $\mathbf{Def}(T_1) \to \mathbf{Def}(T_2)$, then there exists an* *automorphism σ of M such that $g = \sigma f$.*

Proof. Since f and g are surjective, we just need to show that

$$f(u) \mapsto g(u)$$

satisfies $\varphi(f(u)) \iff \varphi(g(u))$ for all tuples $u \in U$ and formulas $\varphi(x)$. This works if the preimage of any 0-definable set in M under f is the same as its preimage under g, and this is precisely the assumption that f^* and g^* induce the same elementary functor. □

Proposition 2.3.19. *An abstract interpretation $F : T_1 \to T_2$ can be realized as a concrete interpretation $(\overline{f}, \overline{f}^*) : M \to N$ for some $M \models T_1$ and $N \models T_2$.*

Proof. By 2.4.1 and the discussion in Remark 2.5.6, given *any* model $N \models T_2$, we can take reducts along the interpretation F and obtain a model $M \models T_1$. □

Proposition 2.3.20. *Let $M \models T$ and $M' \models T'$ be $\aleph_0$-categorical. Then, there exists a strict abstract bi-interpretation between T and T' if and only if there exists a strict concrete bi-interpretation between M and M'.*

Proof. If there exists a strict concrete bi-interpretation of the countable models, then by the previous proposition 2.3.14, the constituent concrete interpretations (f, f^*) and (g, g^*) induce abstract interpretations $F : T \leftrightarrows T' : G$. Since the concrete homotopies are strict, there are definable bijections $GFX \xrightarrow{\eta_X} X$ and $FGY \xrightarrow{\epsilon_Y} Y$ for all $X \in \mathbf{Def}(T)$ and $Y \in \mathbf{Def}(T')$, and these with F and G form an abstract bi-interpretation between T and T'.

Conversely, if we know that T and T' are abstractly strict bi-interpretable, then by Theorem 2.4.1, the abstract strict bi-interpretation $T \simeq T'$ induces a pair of elementary functors $\mathbf{Def}(T) \leftrightarrows \mathbf{Def}(T')$. Since the constituent abstract interpretations of the bi-interpretation are strict, the families of surjective definable functions $\{\eta_X\}$ and

$\{\epsilon_Y\}$ in the definition 2.3.13 of an abstract bi-interpretation must be definable bijections. Now, if we take points in M', by taking reducts we have the data of a strict interpretation $(f, f^*) : M \to M'$, and similarly by taking points in M and taking reducts we have the data of a strict interpretation $(g, g^*) : M' \to M$, and the graphs of the definable bijections $\{\eta_X\}$ and $\{\epsilon_Y\}$ are precisely the definable equalizer relations needed to make $(g * f, g^* f^*) \sim 1_M$ and $(f * g, f^* g^*) \sim 1_{M'}$, which gives a concrete strict bi-interpretation between M and M'. □

Here is an example of a bi-interpretation neither of whose constituent interpretations are invertible.

Definition 2.3.21. Let M be an $\mathcal{L}$-structure. Let $n \geqslant 1$.

We define the n-diagonal interpretation $(f_n, f_n^*) : M \to M$ as follows: write $\Delta_n(M)$ for the diagonal of M^n, which is definable, and put f_n to be the bijection $\Delta_n(M) \simeq M$ by $(m, \ldots, m) \mapsto m$. Then f_n^* pulls back every definable set $X \subseteq M^k$ to the obvious definable subset $f_n^* X \subseteq \Delta_n(M) \times$ (k times) $\times \Delta_n(M)$.

Example 2.3.22. The n-diagonal interpretations $T \xrightarrow{\Delta_n} T$ for $n > 1$ are pseudo-inverse to themselves, but do not admit inverses. This is because if $(g, g^*) : M \to M$ were an inverse to the n-diagonal interpretation (f_n, f_n^*), g^* would need to pull back $\Delta_n(M)$ to M. However, if X is a definable set in the k-sort, then $g^* X$ lives in a k'-sort, where k' is a positive integer multiple of k. Therefore, since $\Delta_n(M)$ lives in the n-sort, and there is no positive integer multiple of n which is 1, there is no inverse interpretation.

In what follows, we always work with abstract interpretations and multisorted languages unless otherwise specified.

2.4 Interpretations as elementary functors

The aim of this subsection is to prove the following theorem, which lets us interchange strict abstract interpretations between theories with elementary functors between their logical categories of definable sets.

Theorem 2.4.1. *$I : \mathcal{L}_1 \to \mathcal{L}_2$ is a strict abstract interpretation $T_1 \to T_2$ if and only if I induces an elementary functor $\mathbf{Def}(T_1) \to \mathbf{Def}(T_2)$.*

Proof. By the above discussion, we just need to show that an elementary functor induces an interpretation. We proceed by an induction on complexity of formulas.

Since finite limits are preserved, I preserves meets of formulas (since the intersection of two subsets of a sort is a pullback).

Since finite sups are preserved, I preserves joins of formulas.

I also preserves negations: $\psi(x)$ and $\neg\psi(x)$ are characterized by their pullback being empty and their sup being all of the ambient sort S_x. Since I preserves pullbacks and finite sups (in particular, the empty sup is the empty set), $I(\psi(x))$ and $I(\neg\psi(x))$ satisfy that their pullback is empty and their join is $I(S_x)$.

I preserves existential quantification since I preserves images and existential quantification is the same as projecting to the sort of the remaining free variables.

Since binding under the existential is the same as projecting to remaining free variables, when we bind all the free variables we are projecting to the empty tuple of variables, which corresponds to the empty product, which is the terminal object 1. So now suppose we have a sentence $\varphi = \exists x \psi(x)$. Since $T \models \exists x \psi(x)$, the image of the corresponding projection to 1 is all of 1. Since I preserves images and terminal objects, the image $I(\exists x \psi(x))$ of the projection $I(\psi(x)) \twoheadrightarrow I(1)$ is again 1, and so $I(\psi(x))$ cannot be the empty subobject 0, since then its unique map to the terminal object would have image 0.

Therefore, I preserves sentences formed by existentially quantifying a formula.

It remains to provide the base for the induction on positive atomic sentences. But these are relations (including equality) evaluated at non-variable terms, say $R(c)$. In $\mathbf{Def}(T_1)$ this is the pullback of $R(x)$ along the inclusion $\{c\} \hookrightarrow S_x$. If $T_1 \models R(c)$, then there is a definable function $1 \to S_x$ picking out c; this factors through the inclusion of $R(x)$ into S_x, giving a pullback square

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & R(x) \arrow[d] \\ 1 \arrow[r] & S_x. \end{tikzcd}$$

Since I preserves finite limits (and hence the terminal object), applying I we get

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & I(R(x)) \arrow[d] \\ 1 \arrow[r] & I(S_x). \end{tikzcd}$$

Since I was a functor, the horizontal map $1 \to I(S_x)$ picks out $I(c)$. Therefore $T_2 \models I(R(c))$, which provides the base for the induction on complexity of formulas and completes the proof. $\square$

Remark 2.4.2. In the previous proof, strictness is needed to even form a functor, because we need the interpretation $I(f)$ of a definable function f to be a T_2-definable function.

However, I being an abstract interpretation of theories and f being a T_1-definable function is not enough to ensure that $I(\Gamma(f))$ is the graph of a T_2-definable function. This is because equality in T_1 may be interpreted as a non-equality equivalence relation E, so that $I(f)$ is only a function after quotienting out its domain and codomain by E. Because of this, T_2 does not necessarily prove that $I(f)$ is a function.

If T_2 contains a quotient for E, then one could try replacing E by the equality relation on that quotient set, which would provide a natural way of replacing the non-strict interpretation I by a strict intepretation homotopic to I. *A priori*, a first-order theory need not contain quotients for all definable equivalence relations. However, we will show in 2.6 that up to abstract bi-interpretability, we can replace any first-order theory T with another first-order theory T^{eq} which *does* contain all quotients for all definable equivalence relations.

2.5 Models as elementary functors

A model M of T an $\mathcal{L}$-theory is an assignment of the symbols of $\mathcal{L}$ onto sets which preserves the truth of sentences: if $T \models \psi$, then $M \models \psi$. **Set** is easily seen to be a logical category, we will see that up to isomorphism of functors, elementary functors $\mathbf{Def}(T) \to \mathbf{Set}$ are precisely the models.

Proposition 2.5.1. *Every model of T corresponds to an elementary functor $\mathbf{Def}(T) \to$ $\mathbf{Set}$.*

Proof. How every model M of T corresponds to a functor $\mathbf{Def}(T) \to \mathbf{Set}$ was described in 2.2.6 ("taking points in M"). That taking points in models preserves finite limits is the content of 2.2.7.

To check preservation of finite sups, let $\{\varphi_1(x), \ldots, \varphi_n(x)\}$ be a finite collection of formulas of the same sort. Then their sup is given by $\bigvee_n \varphi_i(x)$, and the sup of $\{\varphi_1(M), \ldots, \varphi_n(M)\}$ is precisely $\bigcup_n \varphi_i(M)$. The empty sup is the empty formula, represented in $\mathbf{Def}(T)$ by the T-provable equivalence class of "$x \neq x$", and this is interpreted by M as the empty set, which is the empty sup for any set in **Set**.

To check preservation of images, let f be a definable function. The image of f in $\mathbf{Def}(T)$ is just the formula which describes the image of f, and M interprets this formula as the image of $f(M)$.

We have shown that every model M induces a functor, which by an abuse of notation we'll also call M, from $\mathbf{Def}(T) \to \mathbf{Set}$. This completes the proof of the first part of the proposition. Now we'll show that for any elementary functor $F : \mathbf{Def}(T) \to \mathbf{Set}$ is, up to isomorphism of functors, a model.

For every basic sort B, there are canonical isomorphisms $F(B^k) \simeq F(B)^k$. Up to isomorphism of functors (where the isomorphism of functors is given by conjugating by these canonical isomorphisms), we can assume therefore that $F(B^k) = F(B)^k$.

Furthermore, for every sort $\vec{B} = B_1 \times \cdots \times B_n$, there are canonical isomorphisms $F(B_1 \times \cdots \times B_n) \simeq F(B_1) \times \cdots \times F(B_n)$. Again, up to isomorphism of functors, we can assume that $F(\vec{B}) = F(\vec{B})$. Furthermore, if $\varphi(x)$ is a formula of sort B, then there is a canonical definable injection $\varphi(x) \hookrightarrow B$ such that the image of $F(\varphi(x) \hookrightarrow B)$ is a subset of $F(B)$; arguing as before, we can assume up to an isomorphism of functors that $F(\varphi(x)) \subseteq F(B)$. Similarly, we can assume up to an isomorphism of functors that if $T \models \forall x(\varphi(x) \to \psi(x))$, then $F(\varphi(x)) \subseteq F(\psi(x))$.

The canonical isomorphisms described so far induce isomorphisms of Boolean algebras $2^{\vec{B}} \simeq 2^{F(\vec{B})}$. Therefore, up to isomorphism of functors, we can assume that $F(\varphi(x) \vee \psi(x)) = F(\varphi(x)) \cup F(\psi(x))$ (resp. $\wedge$ and negations).

Since F preserves images, then for every definable function f, $F(\mathrm{im}(f)) \simeq \mathrm{im}(F(f))$. Then up to isomorphism of functors, $F(\mathrm{im}(f)) = \mathrm{im}(F(f))$.

Now we have, up to isomorphism, completely "strictified" F. It remains to show that an elementary functor which strictly preserves products, finite sups, and images is a model.

Indeed, let $\vec{c}$ be a tuple of terms such that $R(\vec{c})$ is an atomic sentence. Then by our previous reductions, $F(x = \vec{c}) \subseteq F(R(x))$, so $F \models R(\vec{c})$.

It is obvious that if φ and ψ satisfy that $(T \models \varphi \implies F \models \varphi)$ and $(T \models \psi \implies F \models \psi)$, then $(T \models \varphi \wedge \psi \implies F \models \varphi \wedge \psi)$.

If $\varphi(x)$ is a formula, then $T \models \exists x \varphi(x)$ if and only if the image of the projection of

$\varphi(x)$ to the empty sort (which is the empty product, so is the terminal object 1) is all of 1. Since F is a logical functor, it preserves the terminal object and all maps into the terminal object, so F of the image of the projection of $\varphi(x)$ to the empty sort is still 1. Then $F(\varphi(x))$ cannot be empty, since if it were, the image of its canonical map to 1 would be the empty set. So $F \models \exists x \varphi(x)$.

Similarly, if $T \models \neg\psi$, then if ψ is quantifier-free it is easy to see that $F \models \neg\psi$. If ψ is of the form $\exists \varphi(x)$, then as a subobject of the terminal object 1, $\exists x \varphi(x) = \varnothing$ the empty sup. Since F is logical, it preserves empty sups, so again $\exists x \varphi(x) = \varnothing$ as a subobject of the terminal set 1, and therefore, $F \models \neg \exists x \varphi(x)$.

This concludes the induction on complexity of formulas. □

2.5.1 Elementary embeddings as natural transformations of elementary functors

If elementary functors are models, what do the natural transformations between these elementary functors correspond to at the level of models?

Let us recall the various notions of maps between two $\mathcal{L}$-structures.

Definition 2.5.2. Let M_1 and M_2 be $\mathcal{L}$-structures. An **$\mathcal{L}$-homomorphism** is a $\mathsf{Sorts}(\mathcal{L})$-indexed collection of functions

$$\{\eta_S : M_1(S) \to M_2(S)\}_{S \in \mathsf{Sorts}(\mathcal{L})}$$

which preserve the interpretations of the nonlogical symbols of $\mathcal{L}$ in M_1 and M_2. Remembering our convention that our collections of sorts are closed under formation of finite tuples, we also require, for every finite tuple of sorts $\vec{S} = (S_1, \ldots, S_n)$,

$$\eta_{\vec{S}} = \eta_{S_1} \times \eta_{S_2} \times \cdots \times \eta_{S_n}.$$

Now, "preserving the interpretations of nonlogical symbols in $\mathcal{L}$" means:

1. For each constant $c \in \mathcal{L}$ of sort S, η_S sends $c^{M_1} \mapsto c^{M_2}$,
2. For each relation symbol $R \in \mathcal{L}$ of sort S, for any $\overline{x} \in M_1(S)$, $M_1 \models R^{M_1}(\overline{x}) \implies M_2 \models R^{M_2}(\eta_S(\overline{x}))$.
3. For each function symbol $f \in \mathcal{L}$ of sort $S_1 \to S_2$, whenever $f^{M_1}(\overline{x}) = \overline{y}$, then $f^{M_2}(\eta_{s_1}(\overline{x})) = \eta_{S_2}(\overline{y})$.

An $\mathcal{L}$-homomorphism is called **strict** if it preserves inequality and the complements of the relation symbols. If $\eta : M_1 \to M_2$ is strict, it preserves the truth of all quantifier-free $\mathcal{L}$-formulas in M_1 and M_2: for all quantifier-free $\psi(x_1, \ldots, x_n)$,

$$M_1 \models \psi(a_1, \ldots, a_n) \implies M_2 \models \psi(\eta(a_1), \ldots, \eta(a_n)).$$

If one is able to remove the quantifier-free stipulation above, so that η preserves the truth of *all* $\mathcal{L}$-formulas, then η additionally *reflects* the truth of all $\mathcal{L}$-formulas: for every $\psi(x_1, \ldots, x_n)$,

$$M_1 \models \psi(a_1, \ldots, a_n) \iff M_2 \models \psi(\eta(a_1), \ldots, \eta(a_n)).$$

In this case, η is called an **elementary embedding**. By an easy induction on the complexity of formulas, two models connected by an elementary embedding necessarily have the same theory.

An $\mathcal{L}$-homomorphism $M \to M$ is called an **$\mathcal{L}$-automorphism** if it admits an inverse $\mathcal{L}$-homomorphism; it is easy to see that any $\mathcal{L}$-automorphism is an elementary embedding.

Lemma 2.5.3. *A natural transformation f between models $M_1 \to M_2$ of an $\mathcal{L}$-theory T is precisely an elementary embedding.*

Proof. Since a model is an elementary functor, the components of a natural transformation are induced by restricting its components at all (finite products of) sorts; naturality requires f to send tuples $\overline{x}$ inside a definable set X^{M_1} to inside X^{M_2}. (In particular, natural transformations preserve types: $\mathrm{tp}(x/A) = \mathrm{tp}(f(x)/f(A))$.) Hence (because we have complementation) $M_1 \models \varphi(x) \iff M_2 \models \varphi(f(x))$. □

Remark 2.5.4. More generally, in coherent (i.e. positive existential fragments of first-order) logic, natural transformations are just $\mathcal{L}$-homomorphisms; every finitary first-order theory is bi-interpretable with its Morleyization, which is coherent.

Remark 2.5.5. Now that we have shown that natural transformations between elementary functors correspond to elementary embeddings of the corresponding models, it is clear that the correspondence between elementary functors $\mathbf{Def}(T) \to \mathbf{Set}$ and models $M \models T$ described in 2.5.1 implements an equivalence of categories between:

1. The category of models of T, and
2. the category of strict elementary functors $\mathbf{Def}(T) \to \mathbf{Set}$.

Remark 2.5.6. Since we have shown (Proposition 2.5.1) that models of T are elementary functors $\mathbf{Mod}(T) \to \mathbf{Set}$, and that interpretations $T_1 \to T_2$ are elementary functors $\mathbf{Def}(T_1) \to \mathbf{Def}(T_2)$ (Theorem 2.4.1), it follows that any interpretations $I : T_1 \to T_2$ induces via precomposition a functor

$$I^* : \mathbf{Mod}(T_2) \to \mathbf{Mod}(T_1), \quad \text{by} \quad \big(M : \mathbf{Def}(T_2) \to \mathbf{Set}\big) \mapsto \big(M \circ I : \mathbf{Def}(T_1) \to \mathbf{Def}(T_2) \to \mathbf{Set}\big).$$

Thus, given an interpretation $T_1 \to T_2$, every model of T_2 determines a model of T_1 by "restricting to the image of I". We call such functors between categories of models **reduct functors**. The prototypical example is when the interpretation I is induced by an inclusion of languages; then the reduct functor is literally the reduct to the smaller language.

For the rest of this document, when we say "reduct", we will mean the more general concept of a reduct functor induced by an interpretation.

Remark 2.5.7. Of course, the preceding discussion can be "relativized": instead of working with elementary functors into $\mathbf{Set}$, we could look at all theories interpretable in another theory T, and consider instead of $\mathbf{Mod}(T')$ the category of interpretations $\mathbf{Int}(T', T)$, and the preceding remarks about interpretations inducing "reduct" functors apply equally well.

The analogy carries further: just as natural isomorphisms between strict elementary functors (models) $\mathbf{Def}(T) \to \mathbf{Set}$ correspond precisely with isomorphisms of models, natural isomorphisms between strict elementary functors (abstract interpretations) $F, G\, \mathbf{Def}(T) \to \mathbf{Def}(T')$ correspond precisely to having a strict concrete homotopy in the sense of Ahlbrandt and Ziegler (Definition 2.3.3) between any two concrete interpretations realizing F and G.

2.6 Pretoposes and the $(-)^{\mathrm{eq}}$-construction

One of the key insights in Makkai and Reyes [13] is that when T uniformly eliminates imaginaries, $\mathbf{Def}(T)$ is a small **pretopos**; pretoposes were defined independently by Grothendieck in SGAIV [2] as sites canonically presenting coherent toposes.

Moreover, in [13] it is shown that every logical category $\mathbf{C}$ can be *completed* to a pretopos $\widetilde{\mathbf{C}}$, and this pretopos completion is in a precise sense a categorification of Shelah's $(-)^{\mathrm{eq}}$-construction: $\widetilde{\mathbf{Def}(T)} \simeq \mathbf{Def}(T^{\mathrm{eq}})$.

Remark 2.6.1. Another reason why pretoposes are desirable is that equivalences of the Boolean logical categories $\mathbf{Def}(T_1) \simeq \mathbf{Def}(T_2)$ do not quite correspond to abstract bi-interpretations $T_1 \simeq T_2$, because abstract bi-interpretations are allowed to send sorts to quotients of sorts by definable equivalence relations. For example, if T does not uniformly eliminate imaginaries, $\mathbf{Def}(T)$ is a Boolean logical category but not a pretopos. However, the canonical interpretation $T \to T^{\mathrm{eq}}$ induces an elementary functor $\mathbf{Def}(T) \to \mathbf{Def}(T^{\mathrm{eq}})$. While this canonical interpretation is part of a bi-interpretation, the induced elementary functor between the categories of definable sets cannot be part of an equivalence of categories, because $\mathbf{Def}(T^{\mathrm{eq}})$ has quotients of equivalence relations while by assumption $\mathbf{Def}(T)$ is missing the quotient of some equivalence relation.

However, abstract bi-interpretations *do* correspond to equivalences of categories between the *pretopos completions* of $\mathbf{Def}(T_1)$ and $\mathbf{Def}(T_2)$.

We will recall the $(-)^{\mathrm{eq}}$ construction from model theory. Before we do, we will spell out the notion of being an equivalence relation object in a category.

Definition 2.6.2. An **equivalence relation** (or **internal congruence**) in a category $\mathbf{C}$ with finite limits is the following data:

1. An object X and a subobject $E \hookrightarrow X \times X$,
2. A *reflexivity map* $r : X \to E$ such that r is a section to both projections $\pi_1, \pi_2 : X \times X \to X$,
3. A *symmetry map* $s : E \to E$ such that $\pi_1 \circ s = \pi_2$ and $\pi_2 \circ s = \pi_1$,
4. A *transitivity map* $r : E \times_X E \to E$, where $E \times_X E$ is the pullback of π_1 and π_2, as in the following pullback square (where $i : R \hookrightarrow X \times X$ is the inclusion map):

$$\begin{tikzcd} E \times_X E \arrow[r, "p_2"] \arrow[d, "p_1"'] & R \arrow[d, "\pi_1 \circ i"] \\ R \arrow[r, "\pi_2 \circ i"] & X \end{tikzcd}$$

such that $\pi_1 \circ i \circ p_2 = \pi_1 \circ i \circ t$, and $\pi_2 \circ i \circ p_2 = \pi_2 \circ i \circ t$.

Here is the $(-)^{\mathrm{eq}}$-construction.

Definition 2.6.3. Let T be a complete first-order $\mathcal{L}$-theory. We define the expansion $\mathcal{L}^{\mathrm{eq}}$ of $\mathcal{L}$ as follows: for each φ_E which becomes an internal congruence $E \rightrightarrows X$ in $\mathbf{Def}(T)$, we add a sort S_{φ_E} and a predicate symbol $f_{\varphi_E}(x, e)$, where x is in the sort of x and e is in the sort of S_{φ_E}. The theory T^{eq} is T expanded by sentences which assert that for each φ_E, $f_{\varphi_E}(x, e)$ is the graph of a surjection $X \to S_{\varphi_E}$ which takes each $x \in X$ to its E-class.

Proposition 2.6.4. $\mathbf{Def}(T^{\mathrm{eq}})$ *has finite coproducts.*

Proof. Let Δ_S be the diagonal relation on some sort S of T. Then there is a 0-definable equivalence relation $E_{\Delta_S} \rightrightarrows S \times S$ by $\overline{a} \sim \overline{b} \iff (\overline{a} \in \Delta_S \wedge \overline{a} \in \Delta_S) \vee \left(\overline{a} \in \neg\Delta_S \wedge \overline{b} \in \neg\Delta_S\right)$. Passage to T^{eq} yields two definable constants. Taking binary sequences of these two constants in powers of their imaginary sort $S_{E_{\Delta_S}}$ yields arbitrarily large finite collections of constants, and this lets us take arbitrary finite disjoint unions of definable sets. □

The reason why the $(-)^{\mathrm{eq}}$-construction was introduced was to *eliminate imaginaries.*

Definition 2.6.5. T is said to eliminate imaginaries if for every E-class C of E a definable equivalence relation $E \rightrightarrows X$, there exists a formula $\varphi_C(x, y)$ such that for every model $M \models T$, there exists a tuple b such that b uniquely satisfies $\varphi_C(M_x, b) = C$.

Note that there is a canonical interpretation (which sends equality to equality) of T in T^{eq}.

Proposition 2.6.6. T^{eq} *eliminates imaginaries. Actually, we can do even better:* T^{eq} *will* uniformly *eliminate imaginaries, meaning that we can choose a* $\varphi_E(x, y)$ *instead of one for each* C*.*

Proof. If E is a definable equivalence relation in T, then the graph of f_E uniformly eliminates the imaginaries of E. If E is instead a definable equivalence relation in T^{eq}, it suffices to see that E is equivalent (in the sense of $\mathbf{Def}(T^{\mathrm{eq}})$) to an equivalence already definable in T. Indeed, let $I : T^{\mathrm{eq}} \to T$ be the interpretation defined in the previous remark. Then $I(E)$ is an equivalence relation in T, hence eliminated in T^{eq} by the graph of $f_{I(E)}$. Since $I(E)$-classes are, by definition, compatible with the projections back to the imaginary sorts of the free variables of E, $f_{I(E)}$ definably extends to a definable function whose domain has the same sort as E, and the graph of this eliminates E. $\square$

As the proof of Proposition 2.6.4 demonstrates, if T interprets two constants, then $\mathbf{Def}(T)$ has finite coproducts. We point out another consequence of T interpreting two constants:

Lemma 2.6.7. *If T interprets two constants, then the epimorphisms of $\mathbf{Def}(T)$ are precisely the definable surjections.*

Proof. A definable surjection $f : X \to Y$ is an epimorphism: if f equalizes g_1, g_2 then g_1 and g_2 must agree everywhere on Y.

On the other hand, if f is not surjective and $\{c_1, c_2\}$ are two constants, then f equalizes the maps g_1 and g_2 where g_1 sends all of Y f to c_1 and g_2 sends the image of f to c_1 and $Y \setminus \operatorname{im}(f)$ to c_2, so is not an epimorphism. $\square$

The proof of Proposition 2.6.4 shows that if T uniformly eliminates imaginaries, it interprets two constants. Therefore:

Corollary 2.6.8. *If T uniformly eliminates imaginaries, then the epimorphisms of $\mathbf{Def}(T)$ are precisely the definable surjections.*

Notation 2.6.9. For the remainder of this document, "elimination of imaginaries" will mean uniform elimination of imaginaries in the above sense.

After this section, unless explicitly stated otherwise, we will replace T with T^{eq} if T does not already eliminate imaginaries.

Here are corresponding concepts on the category-theoretic side. Recall that a category $\mathbf{C}$ is said to be **complete** (resp. **finitely complete**) if it has all small limits (resp. finite limits).

Definition 2.6.10. The **kernel pair** of a morphism $f : X \to Y$ in a finitely complete category $\mathbf{C}$ is the internal congruence $\ker(f) \rightrightarrows X$, where the parallel maps are the projections from the pullback $\ker(f) \stackrel{\text{df}}{=} X \times_{f,Y,f} X$.

Definition 2.6.11. A category is **regular** if it is finitely complete and kernel pairs of morphisms admit coequalizers. An epimorphism which arises as the kernel pair of some morphism is called **regular**.

Definition 2.6.12. A category $\mathbf{C}$ is called *Barr-exact* if it is a regular category and all internal congruences in $\mathbf{C}$ are *effective*: they arise as the kernel pair of some morphism. This last condition is the analogue of elimination of imaginaries.

Lemma 2.6.13. $\mathbf{Def}(T)$ *is regular for any first-order theory* T*.*

Proof. Indeed, the kernel pair of a morphism f is coequalized by f', where f' is just f treated as a surjection to $\text{im}(f)$. $\square$

Corollary 2.6.14. *All definable surjections of* T *are regular morphisms in* $\mathbf{Def}(T)$*.*

Definition 2.6.15. A (finitary) *pretopos* is a Barr-exact logical category with finite coproducts.

We give a more direct description, as given in [12].

Definition 2.6.16. A **pretopos** is a category $\mathbf{C}$ satisfying the following:

1. $\mathbf{C}$ has all finite limits (is finitely complete); equivalently, $\mathbf{C}$ has a terminal object and all pullbacks.

2. $\mathbf{C}$ has stable finite sups.

3. $\mathbf{C}$ has stable images.

4. $\mathbf{C}$ has a stable disjoint sum of any pair of objects. A disjoint sum $A \sqcup B$ of objects A, B is a coproduct of A and B such that, for the canonical maps $i : A \hookrightarrow A \sqcup B$ and $j : B \hookrightarrow A \sqcup B$, i and j are monomorphisms and the pullback $A \times_{A \sqcup B} B$ is isomorphic to 0.

Stability for disjoint sums means that whenever we have a diagram of the form

$$\begin{tikzcd}
& A \arrow[r, "i"] & A \sqcup B \\
A' \arrow[ur] \arrow[r] & C' \arrow[ur] & B \arrow[u, "j"'] \\
& B' \arrow[u] \arrow[ur] &
\end{tikzcd}$$

with A' and B' pullbacks, then C' is the disjoint sum of A' and B'.

5. $\mathbf{C}$ has quotients of equivalence relations.

Remark 2.6.17. The only difference between a pretopos and a Boolean logical category is that pretoposes have quotients by all definable equivalence relations. If a quotient by an equivalence relation exists in a logical category, then it is already stable because it is the image of the quotient map and images are stable; by the construction above involving the imaginaries coming from the diagonal and its complement, one has a steady supply of finite disjoint unions of the terminal object, and using these one can form finite disjoint unions of arbitrary objects (easily checked to be stable).

Corollary 2.6.18. $\mathbf{Def}(T^{\text{eq}})$ *is a pretopos.*

Corollary 2.6.19. *The natural notion of a pretopos morphism coincides with elementary functors between logical categories, since disjointness of a coproduct can be checked using a pullback and the empty sup and the property of $\pi : X \to Q$ being a quotient of an equivalence relation $E \subseteq X \times X$ is equivalent to the kernel relation of π (definable from π) being the same as E: elementary functors preserve whatever (fragments of the) pretopos structure are present in a logical category, so in particular preserves all the pretopos structure between two pretoposes.*

Remark 2.6.20. Now that we have introduced the $(-)^{\text{eq}}$-construction, we see that the definition of an abstract bi-interpretation (Definition 2.3.13) can be equivalently defined as a pair of abstract interpretations $F : T \to T'$ and $G : T' \to T$ such that

For any definable set X of T there exists a T^{eq}-definable bijection $\eta_X : X \simeq GF(X)/GF(=)$ (where $GF(=)$ is the definable equivalence relation interpreting equality) such that for any definable function $X \xrightarrow{f} Y$ in T^{eq}, the square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & GF(X)/GF(=) \arrow[d, "GF(g)"] \\ Y \arrow[r, "\eta_Y"'] & GF(Y)/GF(=) \end{tikzcd}$$

commutes, and dually

for any definable set X' of T'^{eq} there exists a definable bijection $\epsilon_{X'} : FG(X')/FG(=) \to X'$ in T'^{eq} such that for any definable function $X' \xrightarrow{f'} Y'$ in T'^{eq}, the square

$$\begin{tikzcd} FG(X')/FG(=) \arrow[r, "\epsilon_{X'}"] \arrow[d, "FG(f')"'] & X' \arrow[d, "f'"] \\ FG(Y')/FG(=) \arrow[r, "\epsilon_{Y'}"'] & Y' \end{tikzcd}$$

commutes.

Thus, to every abstract bi-interpretation of theories, we can associate an equivalence of categories between the pretoposes $\mathbf{Def}(T_1^{\mathrm{eq}}) \simeq \mathbf{Def}(T_2^{\mathrm{eq}})$ (and vice-versa).

This gives a nicer reformulation of Proposition 2.3.20:

Proposition 2.6.21. *Two $\aleph_0$-categorical structures are concretely bi-interpretable if and only if they have abstractly bi-interpretable theories.*

Remark 2.6.22. As remarked in Makkai-Reyes [13], if we change the "finite" in "stability of finite sups" and "finite coproducts" to "small" (in the sense of the ambient universe), we get a Grothendieck topos (c.f. Giraud's theorem 6.1.1 at the beginning of chapter 6.

Notation 2.6.23. For the rest of this document, unless explicitly stated otherwise, we will assume $T = T^{\mathrm{eq}}$, and so $\mathbf{Def}(T)$ will always be a pretopos.

2.7 The 2-category of structures and interpretations

In this section, we form the natural 2-categorical structure of structures and interpretations and study the process of taking endomorphism monoids.

Roughly speaking, a 2-category is a category $\mathbf{C}$ all of whose hom-sets $\mathbf{C}(X, Y)$ are also categories. This means that for any two morphisms $F, G : X \to Y$, there is a notion of a higher "2-morphism" $\eta : F \to G$. The prototypical example for the concept is the category of categories with objects categories and morphisms functors between categories, and with 2-morphisms the natural transformations between functors. For details, we refer the reader to ([9], XII.3).

Definition 2.7.1. A *natural transformation* $\gamma : (f, f^*) \to (g, g^*)$ of two interpretations $A \underset{(g,g^*)}{\overset{(f,f*)}{\rightrightarrows}} B$ is a specification of a 0-definable bijection $f^*(S) \to g^*(S)$ for each sort S of A so that restriction yields 0-definable bijections $f^*X \to g^*X$ for any definable subset of A.

Definition 2.7.2. The 2-*category of first-order structures and interpretations* is given by

$$\mathbf{Struct} \overset{\text{df}}{=} \begin{cases} \underline{\text{Objects}}\text{: first-order structures } A \\ \underline{\text{Morphisms}}\text{: interpretations } (f, f^*) : A \to B \\ \underline{\text{2-morphisms}}\text{: natural transformations.} \end{cases}$$

Proposition 2.7.3. *Let* **TopMon** *be the* 2*-category of topological monoids. There is a contravariant* 2*-functor (which only reverses* 1*-morphisms)*

$$\mathbf{Struct}^{\text{op}} \overset{\text{End}(-)}{\to} \mathbf{TopMon}$$

given by

$$A \mapsto \mathrm{End}(A)$$
$$A \overset{(f,f^*)}{\to} B \mapsto \left(\mathrm{End}(B) \overset{\mathrm{End}((f,f^*))}{\to} \mathrm{End}(A)\right)$$
$$\left((f,f^*) \overset{\gamma}{\to} (g,g^*),\ \textit{for } A \underset{(g,g^*)}{\overset{(f,f^*)}{\rightrightarrows}} B\right) \mapsto \left(\mathrm{End}((f,f^*)) \overset{\mathrm{End}(\gamma)}{\to} \mathrm{End}((g,g^*))\right),$$

where $\mathrm{End}(A)$ *is the monoid of elementary self-maps* $A \to A$ *endowed with the topology of pointwise convergence,* $\mathrm{End}((f,f^*))$ *is induced by restriction (elementarity of an endomorphism ensures this restriction is well-defined) and* $\mathrm{End}(\gamma)$ *is the endomorphism of* A *induced by* γ*, which satisfies (this is the definition of a* 2*-morphism in the* 2*-cat* **TopMon***):*

$$\mathrm{End}(\gamma) \circ \mathrm{End}((f,f^*))(\sigma) = \mathrm{End}((g,g^*))(\sigma) \circ \mathrm{End}(\gamma)$$

for all $\sigma \in \mathrm{End}(B)$.

Proof. This last statement follows from endomorphisms σ being elementary: let x_f be f^*x in f^*A, then

$$\gamma\sigma x_f = \sigma\gamma x_f \implies \gamma \circ (\sigma \restriction f^*A) x_f = (\sigma \restriction g^*A) \circ \gamma x_f$$
$$\implies \mathrm{End}(\gamma) \circ \mathrm{End}(f)(\sigma)(x) = \mathrm{End}(g)(\sigma) \circ \mathrm{End}(\gamma)(x),$$

for all $x \in A$. □

Proposition 2.7.4. *Furthermore, if we discard all morphisms which are not isomorphisms and all natural transformations which are not natural isomorphisms, and thus restrict to the underlying* 2*-groupoid* $\mathrm{core}(\mathbf{Struct})$ *of* **Struct***,* $\mathrm{End}(-)$ *becomes a contravariant* 2*-functor*

$$\mathrm{core}\left(\mathbf{Struct}\right)^{\mathrm{op}} \overset{\mathrm{Aut}(-)}{\to} \mathbf{TopGrp}$$

to the 2*-category of topological groups. In particular, on* 2*-morphisms* $\gamma : (f,f^*) \to (g,g^*)$ *we have* $\mathrm{Aut}(g)(\sigma) = \mathrm{Aut}(\gamma) \circ \mathrm{Aut}(f) \circ \mathrm{Aut}(\gamma)^{-1}$ *for all* $\sigma \in \mathrm{Aut}(B)$.

Remark 2.7.5. Note that $\mathrm{End}(-)$ reflects 2-isomorphisms: if $f \xrightarrow{\gamma} g$ becomes an isomorphism after applying $\mathrm{End}(-)$, then $\mathrm{End}(\gamma)$ is invertible, so γ must have been invertible.

Remark 2.7.6. By the above remark, $\mathrm{End}(-)$ reflects equivalences: if we have a mutual interpretation $f : A \leftrightarrows B : g$ and natural transformations $\eta : \mathrm{id}_A \to gf$ and $\epsilon : fg \to \mathrm{id}_B$ such that $\mathrm{id}_{\mathrm{Aut}(A)} \overset{\mathrm{Aut}(\eta)}{\simeq} \mathrm{Aut}(gf)$ and $\mathrm{id}_{\mathrm{Aut}(B)} \overset{\mathrm{Aut}(\epsilon)}{\simeq} \mathrm{Aut}(fg)$ then η and ϵ must have already been isomorphisms, so that A and B were bi-interpretable.

Remark 2.7.7. $\mathrm{End}(-)$ does not reflect 1-isomorphisms: if we have mutual interpretations $f : A \leftrightarrows B : g$ with $\mathrm{End}(f)$ and $\mathrm{End}(g)$ forming an isomorphism of topological monoids $\mathrm{End}(g) : \mathrm{End}(A) \leftrightarrows \mathrm{End}(B) : \mathrm{End}(f)$, it is not generally true that f and g *invert* each other. This is because there are "homotopies" h in the sense of Ahlbrandt and Ziegler such that $\mathrm{End}(h) = \mathrm{id}$.

2.8 More on $\mathbf{Mod}(T)$

2.8.1 Equivalences of theories induce equivalences of categories of models

Notation 2.8.1. The symbol $\simeq$ between categories means equivalence, not strict isomorphism.

Notation 2.8.2. If $\mathbf{C}$ and $\mathbf{D}$ are categories, we write $[\mathbf{C}, \mathbf{D}]$ for the category of functors $\mathbf{C} \to \mathbf{D}$.

We spell out the purely formal fact that taking functor categories $[-, -]$ preserves equivalences in either argument

Lemma 2.8.3. *Suppose* $\mathbf{C}_1 \simeq \mathbf{C}_2$ *and* $\mathbf{D}_1 \simeq \mathbf{D}_2$. *Then* $[\mathbf{C}_1, \mathbf{D}_1] \simeq [\mathbf{C}_2, \mathbf{D}_2]$.

Proof. Name the functors in the equivalences above $i : \mathbf{C}_1 \simeq \mathbf{C}_2 : j$ and $k : \mathbf{D}_1 \simeq \mathbf{D}_2 : \ell$. Let $F : \mathbf{C}_1 \to \mathbf{D}_2$. We induce a functor $\mathfrak{F} : [\mathbf{C}_1, \mathbf{D}_2] \to [\mathbf{C}_2, \mathbf{D}_2]$ by $F \mapsto kFj$ and $\eta \mapsto k\eta j$ for η a natural transformation. This is clearly functorial, and we'll show it's full, faithful ,and essentially surjective.

Fullness: if $\eta : kFj \to kGj$ is a natural transformation in $[\mathbf{C}_2, \mathbf{D}_2]$, we require an $\eta'' : F \to G$ such that $k\eta''j = \eta$. By the full faithfulness of k, we can lift η to an $\eta' : Fj \to Gj$. So it suffices to show that precomposition by an equivalence is a fully faithful functor between functor categories. To do this, we require the usual construction, requiring the axiom of choice. η' is already a $\mathbf{C}_2$-indexed collection of maps in $\mathbf{D}_1$ between objects in the image of Fj and Gj (which are subsets of the images of F and G, respectively), and we can (non-canonically) use the full faithfulness and essential surjectivity of j to extend η' to an η'' giving a $\mathbf{C}_1$-indexed collection of maps between all objects in the full images of F and G. To be precise: select for each isomorphism class $[b]_\simeq$ of an object $b \in \mathbf{C}_1$ a representative $c_{[b]_\simeq} \in \mathbf{C}_2$, such that $Jc_{[b]_\simeq} \simeq b$, and for each object $b \in \mathbf{C}_1$ an isomorphism $\phi_b : J(c_{[b]_\simeq}) \to b$. Then for all $b \in \mathbf{C}_1$, define $\eta''_b : Fb \to Gb$ by

$$Fb \overset{F\phi_b}{\to} Fjc_{[b]_\simeq} \overset{\eta'_{c_{[b]_\simeq}}}{\to} Gjc_{[b]_\simeq} \overset{G\phi_b^{-1}}{\to} Gb.$$

To see this is a transformation, see that in the naturality diagram

$$\begin{tikzcd}
b \arrow[d, "f"] & Fb \arrow[r, "F\phi_b"] \arrow[d, "Ff"'] & Fjc_{[b]_\simeq} \arrow[r, "\eta'"] \arrow[d] & Gjc_{[b]_\simeq} \arrow[r, "G\phi_b"] \arrow[d] & Gb \arrow[d, "Gf"] \\
b' & b' \arrow[r, "F\phi_{b'}"'] & Fjc_{[b']_\simeq} \arrow[r, "\eta'_{c_{[b']_\simeq}}"'] & Gjc_{[b']_\simeq} \arrow[r, "G\phi_{b'}"'] & Gb'
\end{tikzcd}$$

the squares on the left and right commute by assumption and the center one does as well by the naturality of η'. Hence $\mathfrak{F}$ is full.

Faithfulness: suppose that $\eta \neq \epsilon$ as natural transformations from F to G in $[\mathbf{C}_1, \mathbf{D}_1]$. So there is some $b \in \mathbf{C}_1$ such that $\eta_b \neq \epsilon_b$. By faithfulness of k. $k\eta_b \neq k\epsilon_b$. By the essential surjectivity of j, there is a $c \in \mathbf{C}_2$ such that there is an isomorphism $\phi : jc \simeq b$. Examining the naturality square for η at ϕ yields the identities

$$\epsilon_{jc} = G\phi^{-1} \circ \eta_b \circ F\phi^{-1} \text{ and } \epsilon_{jc} = G\phi^{-1} \circ \epsilon_b \circ F\phi^{-1}.$$

Since functors preserve isomorphisms, and in general isomorphisms $x \to x', y \to y'$ induce a bijection via conjugation $\mathrm{Hom}(x, y) \simeq \mathrm{Hom}(x', y')$, $\eta_b \neq \epsilon_b \implies \eta_{jc} \neq \epsilon_{jc}$. Hence $\mathfrak{F}$ is faithful.

Essential surjectivity: for each functor $H : \mathbf{C}_2 \to \mathbf{D}_2$, we require some $F : \mathbf{C}_1 \to \mathbf{D}_1$ such that there is a natural isomorphism $\mathfrak{F}F = kFj \simeq H$. To do this, we repeat the construction using the axiom of choice from the proof of fullness, this time simultaneously to j and k, so that we have functions $b \mapsto \left(c_{[b]_\simeq}, \phi_b\right)$ and $e \mapsto \left(d_{[e]_\simeq}, \psi_e\right)$. Given a $b \to b'$ in $\mathbf{C}_1$, we construct F via the following sequence of maps:

$$(b \to b') \mapsto \left(\begin{array}{ccc} b & \longrightarrow & jc_{b_\simeq} \\ \big\downarrow & & \\ b' & \longrightarrow & jc_{[b']_\simeq} \end{array}\right) \overset{\text{(f.f.)}}{\mapsto} \left(c_{[b]_\simeq} \to c_{[b']_\simeq}\right)$$

$$\mapsto \left(Hc_{[b]_\simeq} \to Hc_{[b']_\simeq}\right) \mapsto \left(\begin{array}{ccc} Hc_{[b]_\simeq} & \longleftarrow & kd_{\left[Hc_{[b]_\simeq}\right]_\simeq} \\ \big\downarrow & & \\ Hc_{[b']_\simeq} & \longrightarrow & kd_{\left[Hc_{[b']_\simeq}\right]_\simeq} \end{array}\right) \overset{\text{(f.f.)}}{\to} \left(d_{\left[Hc_{[b]_\simeq}\right]_\simeq} \to d_{\left[Hc_{[b']_\simeq}\right]_\simeq}\right),$$

which is easily seen to be functorial. □

Corollary 2.8.4. *If two theories T_1, T_2 are bi-interpretable, then their categories of models are equivalent.*

Proof. A bi-interpretation $T_1 \simeq T_2$ induces an equivalence of pretoposes $\mathbf{Def}(T_1) \simeq \mathbf{Def}(T_2)$. A model of T is just an elementary functor from $\mathbf{Def}(T)$ into $\mathbf{Set}$. Elementary functors are closed under composition, so the restriction of $\mathfrak{F}$ as above to the elementary functor categories is well-defined, and must be an equivalence. □

In light of this fact, it is natural to ask for a converse: if $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$, then is there a bi-interpretation $T \simeq T'$? Can we find a bi-interpretation which induces the original equivalence $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$?

Later, we will give an example which shows that the answer to the first question is "no". The conceptual completeness theorem of Makkai and Reyes [13] says that if we are given an interpretation $T \to T'$ to start with, then if the induced functor $\mathbf{Mod}(T') \to \mathbf{Mod}(T)$ is an equivalence, then the interpretation must have been a bi-interpretation.

2.8.2 Accessibility of $\mathbf{Mod}(T)$

Two important features of the category of models of a theory T are that it has all filtered colimits, and any model can be written as a filtered colimit of elementary submodels the size of the theory.

Proposition 2.8.5. $\mathbf{Mod}(T)$ *has all filtered colimits.*

Proof. Standard inductive construction. □

Proposition 2.8.6. *Let T be a first-order theory. For every $N \in \mathbf{Mod}(T)$, N is either of cardinality $|T|$, or N is the filtered colimit over its elementary submodels of smaller cardinality than N. In fact, N is the filtered colimit over its elementary submodels of cardinality $|T|$.*

Remark 2.8.7. Computing the filtered colimit of a diagram of countable models actually yields that every uncountable model of a first-order theory is the union of a proper infinite elementary chain of submodels:

1. By downward Lowenheim-Skolem, every element x of the uncountable model N is contained in a countable elementary submodel M_x. For cardinality reasons,

there must be at least $|N|$ distinct countable elementary submodels that arise this way.

2. Index the M_x's by α the first ordinal of length $|N|$. By downward Lowenheim-Skolem, amalagate M_0 with M_1, then with M_2, taking the union at ω. Then amalgamate this union with M_ω, etc.; continuing until α, we end up with an elementary chain which covers N.

We record here some consequences of accessibility.

Lemma 2.8.8. *Suppose there is an equivalence $F : \mathbf{Mod}(T) \simeq \mathbf{Mod}(T') : G$. Then for all models $M \models T$ where $M = |T|$, $|F(M)| = \max(|T|, |T'|)$.*

Proof. Write $F(M)$ as a filtered colimit over distinct elementary submodels M_i' of size $|T'|$. Passing through the equivalence, write $M \simeq \varinjlim G(M_i')$. Since M has cardinality $|T|$, only $|T|$-many of the $G(M_i')$s are required in the filtered colimit for M. Therefore, only $|T|$-many of the M_i's are required in the filtered colimit for $F(M)$. Since each M_i' has size $|T'|$, $|F(M)|$ is bounded from above by the size of the $|T|$-indexed disjoint union of the M_i''s, which has size $|T| \times |T'| = \max(|T|, |T'|)$. By the construction in 2.8.7, $F(M)$ is actually a proper elementary chain of length $|T|$ and therefore $|F(M)|$ is at least as big as M_0' plus a single point for every model in the chain, so $|F(M)|$ is at least as big as $|T| + |T'| = \max(|T|, |T'|)$. □

Proposition 2.8.9. *Suppose there is an equivalence $F : \mathbf{Mod}(T) \simeq \mathbf{Mod}(T') : G$. Then for all models $M \models T$, $|F(M)| = \max(|M|, |T'|)$.*

Proof. The proof is the same as that of 2.8.8. □

Corollary 2.8.10. *Let κ be an infinite cardinal. Then for countable theories, κ-categoricity is invariant under bi-interpretation.*

Proof. A bi-interpretation induces an equivalence of categories of models, and by the proposition 2.8.9, this sends models of size κ to models of size κ. Since it is an equivalence, this induces a bijection between the isomorphism classes of models of size κ on either side. $\square$

2.9 $\aleph_0$-categorical structures and theories

In this section, we review the theory of $\aleph_0$-categorical structures and prove some lemmas which will be necessary for our main results. In the rest of the thesis, unless if we say otherwise, an $\aleph_0$-categorical theory will always mean (in light of our convention 2.6.23) the $(-)^{\mathrm{eq}}$ of a one-sorted $\aleph_0$-categorical theory.

2.9.1 The Ryll-Nardzewski theorem

There is a nice description of what the automorphism groups of $\aleph_0$-categorical structures look like. As permutation groups on ω, they must be *oligomorphic*; this is the Ryll-Nardzewski theorem.

Definition 2.9.1. A group action $G \rightharpoonup X$ is *oligomorphic* if each of the product actions

$$G \rightharpoonup X, G \rightharpoonup X^2, G \rightharpoonup X^3 \dots$$

has only finitely many orbits.

Theorem 2.9.2. (Ryll-Nardzewski) *A structure M is $\aleph_0$-categorical if and only if its automorphism group action is oligomorphic.*

Proof. Suppose M is $\aleph_0$-categorical. The omitting types theorem says that a non-isolated type can be omitted, and every infinite compact space must have a non-isolated point. So the type spaces of M in every tuple of sorts have to be finite, and every type is isolated by a formula, so M is ω-saturated. Then any two tuples of the same type are conjugate by an automorphism (via a back-and-forth argument; homogeneity follows from the fact that naming finitely many constants doesn't change the saturation), so $\mathrm{Aut}(M)$ is oligomorphic. Conversely, suppose towards the contrapositive that M was not $\aleph_0$-categorical. Then a type space of M is infinite, since any point of a finite Stone space (whence Hausdorffness) is isolated. The number of types in a tuple of sorts is a lower bound on the number of $\mathrm{Aut}(M)$-orbits on that tuple of sorts, so $\mathrm{Aut}(M)$ is not oligomorphic. □

Here are some examples.

Example 2.9.3. **(i)** Consider the theory of a dense linear order, which at cardinality $\aleph_0$ has just one model: the rationals with the canonical ordering. The orbits in higher powers are determined by how we fiddle with "$<$" and "$=$" relating finitely many points picked from $\mathbb{Q}$.

(ii) A theory with a single equivalence relation with infinitely many infinite classes.

(iii) The theory of equality on an infinite set.

(iv) The theory of the countable random graph.

(v) Relatedly to the above examples: the theory of any Fraïssé limit.

(vi) Here is a nonexample, which we know is not $\aleph_0$-categorical and hence not saturated by Ryll-Nardzewski 2.9.2: $(\mathbb{N}, <)$. This does not realize the type of the point at infinity.

Here is what an ω-saturated extension of this looks like:

$$\mathbb{N} + \dots \mathbb{Z} + \mathbb{Z} + \dots + \mathbb{Z} + \dots$$

where each $\mathbb{Z}$ is equipped with the usual ordering; we can think of those as points at infinity. These copies of $\mathbb{Z}$ are actually *dense*, so the order type properly written is $\mathbb{N} + \mathbb{QZ}$.

(vii) For another non-example which is more purely model-theoretic, take the theory of just equality in an infinite model and name infinitely many distinct constants. Note that if we name just finitely many constants, we still have the Ryll-Nardzewski 2.9.2 theorem—the finitely many orbits are just those constants and then the orbit which contains everything else—but as soon as we name infinitely many, we can take a model which consists of *just* those constants versus a model where we've added an unnamed element.

2.9.2 The Coquand-Ahlbrandt-Ziegler theorem

Definition 2.9.4. Given a group action $G \rightharpoonup M$, we can canonically turn M into a first-order structure $\operatorname{Inv}(G \rightharpoonup M)$, called the *invariant* structure of $G \rightharpoonup M$, in the language where we name every G-invariant subset of any finite power M^n of M with a new predicate symbol.

Theorem 2.9.5. (Coquand, Ahlbrandt-Ziegler, [1]) *Two $\aleph_0$-categorical structures are bi-interpretable if and only if their automorphism groups are isomorphic as topological groups.*

Proof. Let A and B be $\aleph_0$-categorical, and let G_1 and G_2 be their automorphism groups. Suppose there is a topological isomorphism $\varphi : G_1 \xrightarrow{\sim} G_2$.

Since our automorphism groups are topologized under pointwise convergence, open subgroups are stabilizers of tuples. By the Ryll-Nardzewski theorem, there are only finitely many orbits of $G_2 \rightharpoonup B$. Take representatives $\overline{b} \overset{\text{df}}{=} (b_1, \ldots, b_n)$ of those orbits. Consider $\operatorname{Stab}(\overline{b})$ an open subgroup of G_2. This corresponds via the topological isomorphism to an open subgroup H of G_1, which we can assume is of the form $\operatorname{Stab}(\overline{a} \overset{\text{df}}{=} (a_1, \ldots, a_k))$. The domain U of the interpretation will be all the

G_1-conjugates of

$$\bigcup_{i \leqslant n} (a_i, \overline{a})$$

and the interpretation

$$U \overset{f}{\twoheadrightarrow} B$$

is given by

$$f\big(\sigma(a_i), \sigma(a_1), \ldots, \sigma(a_k)\big) \overset{\text{df}}{=} \varphi(\sigma)(b_i)$$

for $1 \leqslant i \leqslant k$ and $\sigma \in G_1$. Carrying out this process for the inverse topological isomorphism $\psi : G_2 \to G_1$ and obtaining a $V \overset{g}{\twoheadrightarrow} B$, we see that (f, f^*) and (g, g^*) form a concrete bi-interpretation $\text{Inv}(G_1) \simeq \text{Inv}(G_2)$. (To take care of the necessary homotopies: since f and g will be bijections $U \simeq B$ and $V \simeq A$ and they were defined by translating orbit representatives, the obvious isomorphisms $g^* f^* B \simeq B$ and $f^* g^* A \simeq A$ gotten by chasing $b \in B$ and $a \in A$ through f and g will be G-invariant.) □

Remark 2.9.6. The conclusion of this theorem fails to hold as soon as we weaken the ω-categoricity assumption, if instead of looking at the topological automorphism group of the unique countable model we look at the topological automorphism group of a countable saturated model.

For example, let T be the theory of an infinite set expanded by countably infinitely many distinct constants. The saturated countable model M of this theory has infinitely many elements which are not constants (these are realizations of the omittable type which says "I am not any of the constants."). The $\text{Aut}(M)$-invariant structure $\text{Inv}(\text{Aut}(M) \rightharpoondown M)$ (see 2.9.4) on M recognizes this omittable type as an infinite predicate which contains no constants.

Since no infinite definable set in $M \models T$ contains no constants, M is not bi-interpretable with $\text{Inv}(\text{Aut}(M) \rightharpoondown M)$.

2.10 Recovering $\mathbf{Mod}(T)$ from $\mathrm{End}(M)$

In this section, we will prove (this appears without proof in a paper [11] by Daniel Lascar):

Proposition 2.10.1. *Let T_1 and T_2 be ω-categorical theories. Let $\mathrm{End}_\omega(-)$ take an ω-categorical theory to the monoid of endomorphisms (in $\mathbf{Mod}(T)$) of its countable model. Then every isomorphism of monoids $F : \mathrm{End}_\omega(T_1) \to \mathrm{End}_\omega(T_2) : G$ induces an equivalence of categories $\overline{F} : \mathbf{Mod}(T_1) \to \mathbf{Mod}(T_2) : \overline{G}$.*

Proof. The functor is obtained by taking colimits of countable submodels. If $N \models T$, we write $\mathrm{Age}_\omega(N)$ for the diagram of countable elementary submodels of N with inclusions between them (these inclusions are automatically elementary maps since the countable models are elementary submodels of N).

$\mathrm{Age}_\omega(N)$ is a filtered diagram in $\mathbf{Mod}(T)$, and $N \simeq \varinjlim(\mathrm{Age}_\omega(N))$.

> *Proof of claim.* Filteredness is equivalent to every finite subdiagram admitting a cocone in the diagram, and this follows from Lowenheim-Skolem: a finite subdiagram in this case is just a finite collection of countable elementary submodels of N. Then N models the elementary diagram of the union of these countable elementary submodels, and so by Lowenheim-Skolem admits a countable elementary submodel which is a cocone to the finite subdiagram.
>
> Since every $n \in N$ is contained in some countable submodel M_n, a cocone $M_n \overset{f_{M_n}}{\to} N'$ under $\mathrm{Age}_\omega(N)$ extends uniquely to a map out of N by sending $n \mapsto f_{M_n}(n)$; the compatibility of the f_{M_n} with the transition maps in the diagram $\mathrm{Age}_\omega(N)$ ensures that this map is well-defined. So N satisfies the universal property of the colimit, hence is isomorphic to the colimit. $\square$

Since every endomorphism of a countable model $M \models T_1$ is an elementary embedding of the form $M \hookrightarrow M$, the isomorphism F of endomorphism monoids tells us how to define $\overline{F}$ on $\mathrm{Age}_\omega(N)$. We extend this to a true functor $\overline{F} : \mathbf{Mod}(T_1) \to \mathbf{Mod}(T_2)$ by defining

$$\overline{F}\,(N_1 \hookrightarrow N_2) \overset{\mathrm{df}}{=} \varinjlim \left(\overline{F}\,\mathrm{Age}_\omega(N_1)\right) \hookrightarrow \varinjlim \left(\overline{F}\,\mathrm{Age}_\omega(N_2)\right),$$

where the induced map is the canonical comparison map between colimits, induced by the natural inclusion of $\mathrm{Age}_\omega(N_1)$ in $\mathrm{Age}_\omega(N_2)$. Functoriality follows from the uniqueness of these comparison maps. $\overline{G} : \mathbf{Mod}(T_2) \to \mathbf{Mod}(T_1)$ is defined entirely analogously.

It now remains to show that $\overline{F}$ and $\overline{G}$ form an equivalence of categories when they are induced by F and G forming an isomorphism of monoids. Since $\overline{G}$ inverts $\overline{F}$ on countable models and elementary embeddings between them, there is already a natural map, in fact a canonical comparison map

$$N \simeq \varinjlim \mathrm{Age}_\omega(N) \longrightarrow \overline{GF}N.$$

To see that this is in fact an isomorphism, it suffices to see that any copies of the countable model $M' \models T_2$ that show up in $\overline{F}(N)$ is in fact of the form $\overline{F}(M)$ for some $M \hookrightarrow N$.

Since filtered colimits in $\mathbf{Mod}(T)$ are unions of the models that appear in the underlying diagram of the filtered colimit, any countable elementary submodel $M' \xrightarrow{i} \overline{F}(N)$ is covered by countably many elementary submodels $\{F(M_m)\}_{m \in M'}$ (since each element of M' is contained in some $F(M_i)$).

By using Lowenheim-Skolem again, we can jointly embed the M_m into another countable elementary submodel $\widetilde{M}$ of N. Then the elementary embedding i factors through the inclusion of the countable elementary submodel $\overline{F}(\widetilde{M})$ into $\overline{F}(N)$. Viewing the map $M' \to \overline{F}(\widetilde{M})$ as an endomorphism $M' \to M'$, we apply the isomorphism to obtain a corresponding endomorphism $\overline{G}(M') \to \widetilde{M}$. Since the composition of elementary embeddings is an elementary embedding, $\overline{G}(M')$ is part of $\mathrm{Age}_\omega(N)$, so that M' of the form $\overline{F}(\overline{G}(M'))$. $\square$

Chapter 3

Ultraproducts, ultracategories, and Makkai's strong conceptual completeness

In this chapter we provide the necessary background on Makkai's theory of ultracategories.

3.1 Introduction

Definition 3.1.1. An **ultraproduct** of a family $(A_i)_{i \in I}$ of non-empty sets with respect to a non-principal ultrafilter $\mathcal{U}$ on I is the set

$$\prod_{i \to \mathcal{U}} A_i \overset{\mathrm{df}}{=} \textstyle\prod_{i \in I} A_i \,/\! \sim_{\mathcal{U}},$$

where $(x_i)_{i \in I} \sim_{\mathcal{U}} (y_i)_{i \in I}$ if and only if $\{j \in I \,|\, x_i = y_i\} \in \mathcal{U}$. Given a representative $(x_i)_{i \in I}$ of a $\mathcal{U}$-class, we write $[x_i]_{i \to \mathcal{U}}$ for its $\sim_{\mathcal{U}}$-class.

Inside the category **Set**, this definition can be recast as the filtered colimit

$$\prod_{i \to \mathcal{U}} A_i \stackrel{\text{df}}{=} \varinjlim \left(\prod_{i \in P} A_i \right)_{P \in \mathcal{U}},$$

where the transition maps in the filtered diagram $\prod_{i \in P} A_i \to \prod_{i \in P'} A_i$ are the projections to the coordinates $P \cap P'$. This correctly handles the possibility that on a $\mathcal{U}$-small set of indices, the sets A_i are empty.

However, it is safe to assume that for the remainder of this document, we will ignore empty models, and so the definition of an ultraproduct of sets can be taken to be the first one.

Definition 3.1.2. When computing an ultraproduct of sets $\prod_{i \to \mathcal{I}} A_i$, we will follow the conventions:

1. Whenever we form an I-indexed product $\prod_{i \in I} A_i$, we will think of each sequence $(a_i)_{i \in I}$ as the set $\{(i, a_i)\}$ and we will always write $\prod_{i \in I} A_i$ as the set of those sequences: $\{\{(i, a_i)\}\}$.
2. Whenever we have a set X and an equivalence relation $E \rightrightarrows X$, we will always write the quotient X/E as the set of (literal) equivalence classes of X.

With these conventions in place, we know *exactly* what set is the ultraproduct of a given family of sets; by applying the ultraproduct construction to the graphs of functions $(X_i \to Y_i)_{i \in I}$, we also know exactly how to take ultraproducts of functions. This all determines **ultraproduct functors** $[\mathcal{U}] : \mathbf{Set}^I \to \mathbf{Set}$, for every I and every ultrafilter $\mathcal{U}$ on I.

Since $\mathbf{Mod}(T)$ is the category of elementary functors (which we think of as pretopos morphisms) $\mathbf{Pretop}(\mathbf{Def}(T), \mathbf{Set})$, once we have specified how to take ultraproducts in **Set**, this tells us how to define ultraproducts of models "pointwise":

Definition 3.1.3. Let $(M_i)_{i \in I}$ be an I-indexed sequence of models of T. We define the **ultraproduct of models** $\prod_{i \to \mathcal{U}} M_i$ to be the following elementary functor $\mathbf{Def}(T) \to$

Set: on objects $A \in \mathbf{Def}(T)$, put

$$\left(\prod_{i \to \mathcal{U}} M_i\right)(A) \stackrel{\text{df}}{=} \prod_{i \to \mathcal{U}} (M_i(A)),$$

so that we have defined where $\prod_{i \to \mathcal{U}} M_i$ sends the object A to be precisely the ultraproduct in **Set** of where each elementary functor M_i sends A.

This determines where $\prod_{i \to \mathcal{U}}(M_i)$ sends maps $f : A \to B$ in $\mathbf{Def}(T)$, by treating f as its graph relation.

As pointed out by Makkai in [12], the content of the Łos theorem (see 3.2.1) is that the previous definition of an ultraproduct of models is still an ultraproduct of models (and this boils down to showing that the ultraproduct functors on **Set** are elementary functors $\mathbf{Set}^I \to \mathbf{Set}$). However, in $\mathbf{Mod}(T)$, ultraproducts of models admit no nice definition in terms of a filtered colimit of infinite products as when we were computing ultraproducts in **Set**—because infinite products of models might not exist.

However, since ultraproducts of models are still computed sort-by-sort (indeed, definable set-by-definable set), one might believe that there is some residual "niceness" from **Set** manifesting in how the ultraproducts of models interact with each other.

The purpose of the notion of ultracategory, modeled after $\mathbf{Mod}(T)$, is to formalize this notion of a category equipped with extra structure coming from a "nice" notion of taking ultraproducts of its objects. In particular, since in **Set**, ultraproducts are a combination of products and filtered colimits, there are purely formal "comparison maps" between ultraproducts arising from the universal properties of products and filtered colimits, and we will see that part of the extra structure includes naming these "comparison maps". Functors which preserve this extra structure are called ultrafunctors, and ultrafunctors between ultracategories $X : \mathbf{K} \to \mathbf{K}'$ will generalize the reduct functors $\mathbf{Mod}(T') \to \mathbf{Mod}(T)$ induced by an interpretation $T \to T'$.

Makkai's duality theorem [12] tells us that there is a dual adjunction between first-order theories (pretoposes) and ultracategories, in fact given by taking appropriate categories of **Set**-valued functors. Strong conceptual completeness says that the counit of this adjunction is an equivalence, i.e. that a pretopos T is equivalent to the category of ultrafunctors $\mathbf{Ult}(\mathbf{Mod}(T), \mathbf{Set})$.

3.2 Basic notions

3.2.1 The Łos theorem

To every non-principal ultrafilter $\mathcal{U}$ on an indexing set I, we have fixed an ultraproduct functor

$$[U] : \mathbf{Set}^I \to \mathbf{Set}, \quad (X_i)_{i \in I} \mapsto \prod_{i \to \mathcal{U}} X_i.$$

The starting point is the Łos ultraproduct theorem, which we rephrase in terms of the ultraproduct functors in **Set**:

Theorem 3.2.1. (Łos theorem) *Let I be an indexing set and U an ultrafilter on I. Then the ultraproduct functor $[U] : \mathbf{Set}^I \to \mathbf{Set}$ is elementary.*

Proof. An elementary functor preserves initial and terminal objects, pullbacks, disjoint sums, and quotients by equivalence relations[1].

- Initial objects: a product of the empty set is the empty set, and a quotient of the empty set is the empty set.
- Terminal objects: a product of terminal objects is terminal, and the quotient of a singleton is a singleton.
- Pullbacks: a product of pullbacks is a pullback, and finite limits commute with filtered colimits.

[1]Note that this implies that images are preserved: the image of any definable function $f : X \to Y$ is the projection to Y of the graph $\Gamma(f)$ of f, and is therefore in definable bijection with the quotient of $\Gamma(f)$ by the definable equivalence relation $(x, y) \simeq (x', y') \iff y = y'$.

- Disjoint sums: a product of disjoint sums is a disjoint sum of products, and colimits commute with colimits.
- Quotients: a product of quotients X_i/E_i is a quotient of products $\prod_I X_i / \prod_I E_i$, and colimits commute with colimits.

$\square$

Corollary 3.2.2. *Let $(M_i)_{i \in I}$ be an I-indexed family of $\mathcal{L}$-structures. For each $\mathcal{L}$-formula $\varphi(x)$, each element $\overline{a}$ of the ultraproduct $\prod_{i \in I} M_i /_{\mathcal{U}}$,*

$$\prod_{i \in I} M_i /_{\mathcal{U}} \models \varphi[\overline{a}] \iff \{ i \in I \mid M_i \models \varphi[a_i] \} \in \mathcal{U}.$$

Proof. By 3.2.1 the ultraproduct functor is elementary, so that the process of taking points inside a model of a definable set commutes with taking ultraproducts. In symbols,

$$\left(\prod_{i \in I} M_i / \mathcal{U} \right)(X) \simeq \prod_{i \in I} M_i(X)/\mathcal{U}.$$

Since this is a filtered colimit, a sequence $\overline{x}$ satisfies that its germ $[\overline{x}]$ is in $\prod_{i \in I} M_i / \mathcal{U}(X)$ if and only if there is some $J \in \mathcal{U}$ such that the restriction of $\overline{x}$ to J is in $\prod_{j \in J} M_j(X)$. i.e. if $x_j \in M_j(X)$ for each $j \in J$. $\square$

We recount the proof via regular ultraproducts of the compactness theorem for first-order logic. This technique will be used in various arguments later on.

Fact 3.2.3. (Compactness theorem for first-order logic) *Let T be a first-order theory. T has a model if and only if every finite subset $T_s \underset{\text{fin}}{\subseteq} T$ has a model.*

Proof. Let I index the finite fragments of T a first-order theory. For each $i \in I$, let P_i be the collection of all $j \in I$ such that viewed as finite fragments of T, $j \supseteq i$. The collection $F \overset{\text{df}}{=} \{P_i\}_{i \in I}$ has the finite intersection property: $P_i \cap P_{i'} = \{ j \in I \mid j \supseteq i \text{ and } j \supseteq i' \} = \{ j \in I \mid j \supseteq i \cup i' \} = P_{i \cup i'}$. Now take a completion $\overline{F}$ of F to an ultrafilter. Let M_i model each finite fragment of T given by $i \in I$, and consider the ultraproduct

$$M^* \stackrel{\text{df}}{=} \prod_I M_i \,/\overline{F}.$$

Then for every sentence $\phi \in T$, ϕ is supported on some subcollection belonging to $\overline{F}$, so is satisfied in M^*. □

We also recount and prove the following useful ultraproduct characterization of elementary classes, due to Chang and Keisler:

Definition 3.2.4. Let M be an $\mathcal{L}$-structure. An *ultraroot* of M is some structure N such that $N^U \simeq M$ for some non-principal ultrafilter $\mathcal{U}$.

Fact 3.2.5. *A class* $\mathbf{C}$ *of* $\mathcal{L}$*-structures is an elementary class if and only if it is closed under isomorphisms, ultraproducts, and ultraroots.*

Proof. Suppose that $\mathbf{C}$ is the objects $\mathbf{Mod}(T)_0$ of $\mathbf{Mod}(T)$ for some $\mathcal{L}$-theory T. Then it closed under isomorphisms, ultraproducts (by the Łos theorem 3.2.1), and ultraroots (since diagonal embeddings into ultrapowers are elementary).

On the other hand, suppose that $\mathbf{C}$ is a class of $\mathcal{L}$-structures closed under isomorphisms, ultraproducts, and ultraroots. Let T be the theory

$$T \stackrel{\text{df}}{=} \bigcap_{M \in \mathbf{C}} \mathbf{Th}_{\mathcal{L}}(M).$$

It suffices to show that $\mathbf{C} = \mathbf{Mod}(T)_0$. By definition, $\mathbf{C} \subseteq \mathbf{Mod}(T)_0$, and the inclusion $\mathbf{C} \hookrightarrow \mathbf{Mod}(T)_0$ reflects isomorphisms. By the Keisler-Shelah isomorphism theorem 3.2.6, the inclusion reflects elementary equivalences. Therefore, if there is an $M \in \mathbf{Mod}(T)_0 \backslash \mathbf{C}$, its theory must not show up in $\{\mathbf{Th}(N) \,|\, N \in \mathbf{C}\}$.

Since $T = \bigcap \{\mathbf{Th}(N) \,|\, N \in \mathbf{C}\}$, for every finite fragment $\Sigma \underset{\text{fin}}{\subseteq} \mathbf{Th}(M)$, there exists an $N_\Sigma \in \mathbf{C}$ such that $\mathbf{Th}(N) \models \Sigma$. (Otherwise, there is a sentence $\psi \in \mathbf{Th}(M) \backslash T$ such that for all $N \in \mathbf{C}$, $N \models \neg\psi$, so that $\neg\psi \in T$, a contradiction).

There is a regular ultrafilter $\mathcal{U}$ such that $\prod_{\substack{\Sigma \to \mathcal{U} \\ \Sigma \subseteq_{\text{fin}} \mathbf{Th}(M)}} N_\Sigma \models \mathbf{Th}(M)$. Since $\mathbf{C}$ was closed under ultraproducts, this contradicts our assumption that $M \notin \mathbf{C}$. Therefore, $\mathbf{C} = \mathbf{Mod}(T)$ is an elementary class.

□

Finally, we state the Keisler-Shelah isomorphism theorem (though much of the time, special models arguments suffice to replace it.)

Theorem 3.2.6. *(Keisler-Shelah isomorphism theorem) Two $\mathcal{L}$-structures are elementarily equivalent if and only if they have isomorphic ultrapowers.*

3.2.2 Frayne's lemma and Scott's lemma

In this subsection, we state Frayne's lemma and the related Scott's lemma, which will be needed for some later results. We omit the proofs (somewhat-elaborate regular ultraproduct arguments) and refer the interested reader to [3].

Lemma 3.2.7. *Let $N \equiv M$ be elementarily equivalent. Then N elementarily embeds into some ultrapower $M^{\mathcal{U}}$ of M.*

Lemma 3.2.8. *Let $M \xrightarrow{f} N$ be an elementary map. Then there is an ultrapower $M^{\mathcal{U}}$ of M and an elementary embedding $N \xrightarrow{g} M^{\mathcal{U}}$ such that the diagram*

$$\begin{tikzcd} M^{\mathcal{U}} & \\ M \arrow[u, "\Delta_M"] \arrow[r, "f"'] & N \arrow[ul, "g"'] \end{tikzcd}$$

commutes.

3.2.3 The Beth definability theorem

In this subsection, we state and prove a version of the Beth definability theorem, due to Bradd Hart.

Theorem 3.2.9. *Let $L_0 \subseteq L_1$ be two languages, so that L_1 has no new sorts. Let T_1 be a theory in L_1. Let F be the reduct functor*

$$\mathbf{Mod}(T_1) \to \mathbf{Mod}(\varnothing_{L_0})$$

(here $\varnothing_{L_0}$ is the empty theory in L_0, whose models are just all the L_0-structures.)

Suppose that we know any one of the following:

1. *There is a theory T_0 in L_0 such that F factors through $\mathbf{Mod}(T_0)$ with $\mathbf{Mod}(T_1) \to \mathbf{Mod}(T_0)$ an equivalence of categories.*
2. *F is fully faithful.*
3. *F is injective on objects.*
4. *F is full and faithful just on automorphism groups.*
5. *For all $M \in \mathbf{Mod}(T_1)$, every L_0-elementary map $f : F(M) \to F(M)^{\mathcal{U}}$ is (uniquely lifts to) an L_1-homomorphism $f = \widetilde{f} : M \to M^{\mathcal{U}}$ (between M and $M^{\mathcal{U}}$ viewed as L_1-structures.)*

Then: every L_1-formula is T_1-equivalent to an L_0-formula.

Proof. 1 clearly implies 2.

2 is equivalent to 3: assume not 3. Then there are two distinct L_1-expansions M and N of the same L_0-structure K, and so the identity automorphism is not in the image of F restricted to $\mathrm{Hom}_{L_1}(M, N)$: M and N being different must be witnessed by a single tuple k and some symbol R from $L_1 \backslash L_0$ such that $\models R^M(k)$ and $\models \neg R^M(k)$.

Therefore, any automorphism of K which fixes k cannot be L_1-elementary, and this negates 2.

Now assume not 2. Since taking reducts along an inclusion of languages is always faithful, this must fail to be full, and so this must be witnessed by an L_0-elementary, L_1-not-elementary map $g : FM \to FN$. But then the pushforward L_1-structure of M along g induces an L_1-expansion of FN distinct from N, which negates 3.

2 clearly implies 4.

4 implies 5: it suffices to show fullness, so let $f : FM \to (FM)^{\mathcal{U}}$. We use a special models argument: by repeatedly invoking Scott's lemma, start with $FM \to (FM)^{\mathcal{U}}$ and obtain a diagram of iterated ultrapowers

$$\begin{tikzcd}
FM \arrow[r] \arrow[d, "f"'] & (FM)^{\mathcal{U}_0} \arrow[r] \arrow[d] & \left((FM)^{\mathcal{U}\prime}\right)^{\mathcal{U}_2} \arrow[r] \arrow[d] & \cdots \\
(FM)^{\mathcal{U}} \arrow[r] \arrow[ur] & \left((FM)^{\mathcal{U}}\right)^{\mathcal{U}_1} \arrow[r] \arrow[ur] & \left(\left(F(M)^{\mathcal{U}}\right)^{\mathcal{U}_1}\right)^{\mathcal{U}_3} \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}$$

so that the vertical arrows become an isomorphism $f_\omega : \mathbf{M}_1 \to \mathbf{M}_2$ in the limit which extends f. The diagonal arrows become an isomorphism $g_\omega \mathbf{M}_2 \to \mathbf{M}_1$ in the limit, and from the commutativity of the diagram at every stage, $g_\omega \circ f_\omega$ becomes an automorphism of $\mathbf{M}_1$ which extends f. Then $g_\omega \circ f_\omega$ lifts to an L_1-automorphism. Restricting $g_\omega \circ f_\omega$ from $\mathbf{M}_1$ to FM, we get that f also lifts to an L_1-homomorphism.

Now, to show that 5 implies that every L_1-formula is equivalent modulo T to an L_0-formula: suppose not, so that there is an L_1 formula $\psi(x)$ such that for all L_0-formulas $\psi(x)$, there exists an a and a b such that $\varphi(a) \wedge \varphi(b)$ for all $\varphi(x) \in L_0$, but $\psi(a) \wedge \neg\psi(b)$.

This is saying that there exists a model M of T_1 on which the indicator function of $\psi(x)$ disagrees with the indicator functions of every $\varphi(x) \in L_0$.

Now, since a and b have the same L_0-type, there exists an ultrafilter $\mathcal{U}$ and a pair of maps

$$FM \underset{\Delta_{FM}}{\overset{f}{\rightrightarrows}} (FM)^{\mathcal{U}}$$

where f is some embedding which sends $b \mapsto a$.

Since b and a have distinct L_1-types, $\widetilde{f}$ is not L_1-elementary. Since $\widetilde{f}$ is a homomorphism lifting an elementary embedding, it is an embedding, so a and b have the same quantifier-free L_1-type. Therefore, ψ was not quantifier free.

This implies that every quantifier-free L_1 formula ψ is T-provably equivalent to an L_0-formula.

Since interpretations commute with quantification, we conclude that every L_1-formula is T-provably equivalent to an L_0-formula. □

With a little more work, we can remove the stipulation that no new sorts are added.

Theorem 3.2.10. *Let $L_0 \subseteq L_1$ be an inclusion of languages, possibly with new sorts. Let T be an L_1-theory.*

Suppose that whenever $M \models T$ and $\mathcal{U}$ is an ultrafilter, then every elementary map

$$M \restriction_{L_0} \to \left(M \restriction_{L_0}\right)^{\mathcal{U}}$$

has a unique lift to a homomorphism on M.

Then every definable set X of T is T-provably equivalent (i.e. equivalent modulo T) to an L_0-imaginary sort.

Proof. The proof proceeds via the following steps:

1. Show that under our assumptions, in every model $M \models T$, the points in M of every L_1-sort is contained in the definable closure of the points in M of the L_0-sorts.

2. By a compactness argument (see proof of 3.2.11), every L_1-sort is the surjective image of an L_0-imaginary sort.

We will spell out the first item, and trust that the reader will be able to imitate the identical compactness argument from 3.2.11. Suppose towards the contrapositive that there exists a model $M \models T$ such that there is an L_1-sort S such that $M(S)$ is not in the definable closure of $M(L_0)$. Then (possibly enlarging M) there are two points $x, y \in M(S)$ with the same type over $M(L_0)$. So, for some ultrapower $M^{\mathcal{U}}$ of M, $\Delta_M(x)$ and $\Delta_M(y)$ are conjugate by an automorphism σ fixing $M(L_0)$.

Then both $\Delta_M : M \to M^{\mathcal{U}}$ and $\sigma \circ \Delta_M : M \to M^{\mathcal{U}}$ lift $\Delta : M \restriction_{L_0} \to \left(M \restriction_{L_0}\right)^{\mathcal{U}}$, which violates the assumptions of the theorem. □

3.2.4 Conceptual completeness

In this subsection, we state the conceptual completeness theorem (7.1.8, [13]) from Makkai-Reyes and give a proof, following (4.4, [8]).

Theorem 3.2.11. *Let T_1 and T_2 eliminate imaginaries. Let $J : \mathbf{Def}(T_1) \to \mathbf{Def}(T_2)$ be an interpretation of T_1 in T_2. Let $J^* \overset{\text{df}}{=} (- \circ J)$ be the induced functor $\mathbf{Mod}(T_2) \to \mathbf{Mod}(T_1)$.*

Then J is an equivalence of categories if and only if J^* is.

Proof. That J^* is an equivalence of categories if J is is purely formal, c.f. the lemma 2.8.3.

Towards the other direction, suppose J^* is an equivalence of categories. We need to show that J is full, faithful, and essentially surjective.

To see that J is faithful: if $f_1 \neq f_2$ for $Y_1 \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y_2$ in $\mathbf{Def}(T_1)$, then their equalizer is not all of Y_1, which is to say that

$$\models \exists y \in Y_1 \ \text{ s.t. } y \notin \mathrm{eq}(f_1, f_2).$$

Since J is an interpretation, as a functor it preserves finite limits, complementation, and existential quantification. Applying J to the above sentence, conclude that $J(f_1) \neq J(f_2)$.

Claim. If J is essentially surjective, it is full.

> *Proof of claim.* If $g : J(Y_1) \to J(Y_2)$ is a definable function, then its graph $\Gamma(g)$ is a definable set $\Gamma(g) \hookrightarrow J(Y_1 \times Y_2)$. If J is essentially surjective, then there is a corresponding $\Gamma(\overline{g}) \hookrightarrow (Y_1 \times Y_2)$ such that $J(\overline{g}) = g$. $\square$

So, it suffices to see that J is essentially surjective.

First, we show that to prove this, it suffices to be able to place every object of $\mathbf{Def}(T_2)$ inside an object coming from T_1:

Claim. Let $X \in \mathbf{Def}(T_2)$. If there exists $Y \in \mathbf{Def}(T_1)$ with $X \hookrightarrow J(Y)$, then there exists $\overline{X} \in \mathbf{Def}(T_1)$ with $J(\overline{X}) = X$.

> *Proof of claim.* Let M and N be two models of T_2. If $J^*M = J^*N$, then $M(X) = N(X)$ since $M(X) \hookrightarrow M(Y) = N(Y) \hookleftarrow N(X)$ and since J^* is an equivalence (consider a lift of the identity and the corresponding naturality square for the inclusion $X \hookrightarrow J(Y)$), $M(X) = N(X)$. $\square$

Next, we claim that if M_2 is any model of T_2, then any element a of M_2 is definable over J^*M_2. Indeed, we can replace M_2 with a larger model such that there are two elements a and b which are not definable over J^*M_2 but which have the same type over J^*M_2. Then there is an ultrapower ${}^*M^2$ of M^2 and an automorphism of this ultrapower which moves $\Delta(a)$ to $\Delta(b)$. This would yield two different embeddings of M_2 in ${}^*M_2$; these agree on J^*M_2, which would contradict that J^* was an equivalence.

We will now use a compactness argument to show that, in $\mathbf{Def}(T_2)$, any definable set Y of T_2 is the image of a definable map from a definable set $J(X)$ coming from T_1.

So, suppose that Y is not covered by any finite collection of functions whose domains lie in sorts coming from T_1. This means that for any finite collection of such functions, every model realizes a witness $d \in Y$ which lies outside the images of the functions. That is, after introducing a generic constant symbol d, the theory

$$T' \overset{\text{df}}{=} T_2 \cup \{\neg\exists r \phi(r,d) \,|\, \phi \text{ is a function whose domain lies in a } T_1\text{-sort}\} \cup \{d \in Y\}$$

is finitely consistent, therefore consistent. So T' has a model. But in any model M of T', the realization of d will not be T_2-provably definable over J^*M (since otherwise the formula $\varphi(x,y)$ which witnesses this can be restricted to a definable function whose domain is in a sort coming from T_1), contradicting the previous claim. Therefore, there exists some definable set $J(X)$ such that there is a definable surjection $J(X) \twoheadrightarrow Y$.

By the second claim of this proof, the kernel relation of the definable surjection $J(X) \twoheadrightarrow Y$ is in the image of J. Therefore, Y is isomorphic to an imaginary sort of T_1, and since T_1 eliminates imaginaries, Y is in the essential image of J. $\square$

3.3 Ultracategories and ultrafunctors

3.3.1 Pre-ultracategories and pre-ultrafunctors

Definition 3.3.1. ([12], Section 1) A **pre-ultracategory** $\underline{\mathbf{S}}$ is a category $\mathbf{S}$ along with specified ultraproduct functors $[U] : \mathbf{S}^I \to \mathbf{S}$ for every set I and every non-principal ultrafilter $\mathcal{U}$ on I.

(Of course, this is not enough structure to nail down what it means to have a nice notion of being able to form ultraproducts of families of objects; there are no restrictions on what these ultraproduct functors might be, or how they interact. For example, given an pre-ultracategory, we could replace $[\mathcal{U}]$ for each I with $[\mathcal{V}]$ for $\mathcal{V}$ some fixed principal ultrafilter, and this would still be a pre-ultracategory.)

The prototypical pre-ultracategory is **Set**; we have already described its ultraproduct functors.

There is an obvious notion of a structure-preserving map between pre-ultracategories.

Definition 3.3.2. ([12], Section 1) A *pre-ultrafunctor* $\underline{\mathbf{S}} \to \underline{\mathbf{S}}'$ is a functor $X : \mathbf{S} \to \mathbf{S}'$ along with a specified *transition isomorphism*

$$\Phi_{X,U} : X \circ [U] \xrightarrow{\simeq} [U] \circ X^I,$$

for each I and each U an ultrafilter on I. That is, we require all diagrams

$$\begin{tikzcd} \mathbf{S}^I \arrow[r, "X^I"] \arrow[d, "{[U]_{\mathbf{S}}}"'] & (\mathbf{S}')^I \arrow[d, "{[U]_{\mathbf{S}'}}"] \\ \mathbf{S} \arrow[r, "X"'] & \mathbf{S}' \end{tikzcd}$$

to commute, where U ranges over all non-principal ultrafilters on I ranging over all small indexing sets. ("Ultraproducts are preserved up to the transition isomorphism $\Phi_{X,U}$.")

Remark 3.3.3. Every functor of points $\mathrm{ev}_{\varphi(x)}$ can be canonically viewed as a pre-ultrafunctor with the transition isomorphisms Φ just the identity maps (corresponding to the equality signs in the above diagrams).

Remark 3.3.4. Because we only require our pre-ultrafunctors to commute with ultraproducts up to transition isomorphisms, one can have functors $X : \mathbf{Mod}(T) \to \mathbf{Set}$ induced by taking certain (clearly non-definable) subsets of models which are isomorphic anyway by *some* natural transformation of functors to a definable functor. We give an example below, which is the basis of the constructions in 8.1.

Example 3.3.5. Let T be the theory of equality on an infinite set expanded by denumerably many distinct constant symbols $\{c_i\}_{i \in \omega}$. Then the functor $X : \mathbf{Mod}(T) \to \mathbf{Set}$ which is induced by sending

$$M \quad \mapsto \quad \{c_i \,\big|\, i \text{ even}\} \cup (M \backslash \{c_i\}_{i \imath \omega})$$

is isomorphic to the functor $\mathrm{ev}_=$, which just takes the 1-sort of any model. The isomorphism $X \simeq \mathrm{ev}_=$ is given by on each model M by making it the identity on the omittable type $(M \backslash \{c_i\}_{i \imath \omega})$; on constants, we use any bijection $\mathbb{N} \to 2\mathbb{N}$, say $k \mapsto 2 \cdot k$.

We want to work with a category of pre-ultrafunctors, so we must describe what it means to have a morphism of pre-ultrafunctors.

Definition 3.3.6. ([12], Section 1) Given two pre-ultrafunctors (X, Φ) and (X', Φ'), we define a map between them, called an **ultratransformation**, to be a natural transformation $\eta : X \to X'$ which satisfies the following additional property: all diagrams

$$\begin{tikzcd} X\left(\prod_{i \to \mathcal{U}} M_i\right) \arrow[r, "\Phi_{(M_i)}"] \arrow[d, "\eta_{\prod_{i \to \mathcal{U}} M_i}"'] & \prod_{i \to \mathcal{U}} X(M_i) \arrow[d, "\prod_{i \to \mathcal{U}} \eta_{M_i}"] \\ X'\left(\prod_{i \to \mathcal{U}} M_i\right) \arrow[r, "\Phi'_{(M_i)}"'] & \prod_{i \to \mathcal{U}} X'(M_i) \end{tikzcd}$$

must commute.

Definition 3.3.7. The **category of pre-ultrafunctors** $\mathbf{PUlt}(\mathbf{Mod}(T), \mathbf{Set})$ comprises the following data:

$$\mathbf{PUlt}(\mathbf{Mod}(T), \mathbf{Set}) \overset{\mathrm{df}}{=} \begin{cases} \text{Objects: pre-ultrafunctors } (X, \Phi) : \mathbf{Mod}(T) \to \mathbf{Set} \\ \text{Morphisms: ultratransformations } \eta : (X, \Phi) \to (X, \Phi'). \end{cases}$$

3.3.2 Ultramorphisms

In **Set** ultraproducts are computed as certain filtered colimits, and so there are canonical comparison maps between them induced by maps between their underlying diagrams (via their universal properties).

For example, consider the terminal map $I \to 1$. This induces a diagonal map $A \to \prod_{i \in I} A$ by $a \mapsto (a, a, a, \ldots)$, and this induced map extends along ultraproducts to give the diagonal map

$$\Delta : M \hookrightarrow M^{\mathcal{U}}$$

of a model into its ultrapower.

In $\mathbf{Mod}(T)$, these filtered colimits don't usually exist because products of models of a first-order theory don't usually exist. For example, $\mathbf{Mod}(T)$ doesn't see the diagonal map $A \to \prod_{i \in I} A$, only the diagonal map $\Delta : M \hookrightarrow M^{\mathcal{U}}$ it induces on models. So the pure category $\mathbf{Mod}(T)$ does not distinguish (say) $\Delta : M \hookrightarrow M^{\mathcal{U}}$ from any other embedding $M \hookrightarrow M^{\mathcal{U}}$, because there is no canonical way to obtain Δ. But once we force $\mathbf{Mod}(T)$ to remember that ultraproducts of models are computed as certain filtered colimits in **Set**, then $\Delta : M \hookrightarrow M^{\mathcal{U}}$ is distinguished by the ultracategory $\mathbf{Mod}(T)$ because it arises in a canonical way.

The purpose of the notion of ultramorphisms is to name all the maps between ultraproducts in an pre-ultracategory which "should" arise in a canonical way. It turns out that this is enough to correct for the laxness in the definition of a pre-ultracategory: an ultracategory will be precisely a pre-ultracategory with as many ultramorphisms as possible, and after that we get Makkai's duality and strong conceptual completeness.

The definition of an ultramorphism

Definition 3.3.8. ([12], Section 3) An **ultragraph** Γ comprises:

(i) Two disjoint sets Γ^f and Γ^b, called the sets of **free nodes** and **bound nodes**, respectively.

(ii) For any pair $\gamma, \gamma' \in \Gamma$, there exists a set $E(\gamma, \gamma')$ of **edges**. This gives the data of a directed graph.

(iii) For any bound node $\beta \in \Gamma^b$, we assign a triple ("ultraproduct data") $\langle I, \mathcal{U}, g \rangle \overset{\text{df}}{=} \langle I_\beta, \mathcal{U}_\beta, g_\beta \rangle$ where $\mathcal{U}$ is an ultrafilter on I and g is a function $g : I \to \Gamma^f$.

Definition 3.3.9. ([12], Section 3) An **ultradiagram of type** Γ in a pre-ultracategory $\underline{\mathbf{S}}$ is a diagram $A : \Gamma \to \mathbf{S}$ assigning an object A to each node $\gamma \in X$, and assigning a morphism in $\mathbf{S}$ to each edge $e \in E(\gamma, \gamma')$, such that

$$A(\beta) = \prod_{i \in I_\beta} A(g_\beta(i))/\mathcal{U}_\beta$$

for all bound nodes $\beta \in \Gamma^b$.

Given this notion of a diagram with extra structure, there is an obvious notion of natural transformations between such diagrams which preserve the extra given structure.

Definition 3.3.10. ([12], Section 3) Let $A, B : \Gamma \to \mathbf{S}$. A **morphism** of ultradiagrams $\Phi : A \to B$ is a natural transformation Φ satisfying

$$\Phi_\beta = \prod_{i \to \mathcal{U}_\beta} \Phi_{g_\beta(i)}$$

for all bound nodes $\beta \in \Gamma^b$.

Finally, we can define ultramorphisms.

Definition 3.3.11. ([12], Section 3) Let $\mathrm{Hom}(\Gamma, \underline{\mathbf{S}})$ be the category of all ultradiagrams of type Γ inside $\underline{\mathbf{S}}$ with morphisms the ultradiagram morphisms 3.3.10 defined above. Any two nodes $k, \ell \in \Gamma$ define *evaluation functors* $(k), (\ell) : \mathrm{Hom}(\Gamma, \underline{\mathbf{S}}) \rightrightarrows \mathbf{S}$, by

$$(k)\left(A \xrightarrow{\Phi} B\right) = A(k) \xrightarrow{\Phi_k} B(k)$$

(resp. ℓ).

An **ultramorphism** of type $\langle \Gamma, k, \ell \rangle$ in $\underline{\mathbf{S}}$ is a natural transformation $\delta : (k) \to (\ell)$.[2]

Examples of ultramorphisms

Let us unravel this definition for the prototypical example $\Delta : M \hookrightarrow M^{\mathcal{U}}$ of an ultramorphism.

[2]Note that in our terminology, an ultramorphism, singular, refers to a collection of possibly many maps (the components of the natural transformation $(k) \to (\ell)$).

Example 3.3.12. Given an ultrafilter $\mathcal{U}$ on I, put:

- $\Gamma^f = \{k\}$,
- $\Gamma^b = \{\ell\}$,
- $E(\gamma, \gamma') = \varnothing$ for all $\gamma, \gamma' \in \Gamma$,
- $\langle I_\ell, \mathcal{U}_\ell, g_\ell \rangle = \langle I, \mathcal{U}, g \rangle$ where g is the constant map to k from I.

By the ultradiagram condition 3.3.9, an ultradiagram A of type Γ in $\underline{\mathbf{S}}$ is determined by $A(k)$, with $A(\ell) = A(k)^{\mathcal{U}}$.

By the ultradiagram morphism condition 3.3.10, an ultramorphism of type $\langle \Gamma, k, \ell \rangle$ must be a collection of maps $\left(\delta_M : M \to M^{\mathcal{U}}\right)_{M \in \mathbf{Mod}(T)}$ which make all squares of the form

$$\begin{array}{ccc} M^{\mathcal{U}} & \xrightarrow{f^{\mathcal{U}}} & N^{\mathcal{U}} \\ {\scriptstyle \Delta_M}\uparrow & & \uparrow{\scriptstyle \Delta_N} \\ M & \xrightarrow[f]{} & N \end{array}$$

commute. It is easy to check that setting $\delta_M = \Delta_M$ the diagonal embedding gives an ultramorphism.

Definition 3.3.13. The next least complicated example of an ultramorphism are the **generalized diagonal embeddings**. Here is how they arise: let $g : I \to J$ be a function between two indexing sets I and J. g induces a pushforward map $g_* : \beta I \to \beta J$ between the spaces of ultrafilters on I and J, by $g_* \mathcal{U} \overset{\text{df}}{=} \{P \subseteq J \,\big|\, g^{-1}(P) \in \mathcal{U}$. Fix $\mathcal{U} \in \beta I$ and put $\mathcal{V} \overset{\text{df}}{=} g_* \mathcal{U}$. Let $(M_j)_{j \in J}$ be a J-indexed family of models.

Then there is a canonical "fiberwise diagonal embedding"

$$\Delta_g : \prod_{j \to \mathcal{V}} M_j \to \prod_{i \to \mathcal{U}} M_{g(i)}$$

given on $[a_j]_{j \to \mathcal{V}}$ by replacing each entry a_j with $g^{-1}(\{a_j\})$-many copies of itself.

In terms of the definition 3.3.11 of an ultramorphism, the free nodes are J, and there are two bound nodes k and ℓ. To k we assign the triple $\langle J, \mathcal{V}, \mathrm{id}_J \rangle$ and to ℓ we assign the triple $\langle I, \mathcal{U}, g\rangle$. Then Δ_g induces an ultramorphism $(k) \to (\ell)$.

3.3.3 What it means for a pre-ultrafunctor to preserve an ultramorphism

Given the protytpical diagonal embedding ultramorphisms Δ_M, we can say what it means that a pre-ultrafunctor (X, Φ) *preserves* diagonal embeddings.

Definition 3.3.14. We say that a pre-ultrafunctor 3.3.2 (X, Φ) is a **Δ-functor** if for every I, for every $\mathcal{U}$, and for every M and the diagonal embedding $M \xrightarrow{\Delta_M} M^{\mathcal{U}}$, the diagram

$$\begin{tikzcd} & X\left(M^{\mathcal{U}}\right) \arrow[dd, "\Phi_{(M)}"] \\ X(M) \arrow[ur, "X(\Delta_M)"] \arrow[dr, "\Delta_{X(M)}"'] & \\ & X(M)^{\mathcal{U}} \end{tikzcd}$$

commutes.

Analogously, we can define what it means for (X, Φ) to preserve a general ultramorphism 3.3.11. Let $(X, \Phi) : \underline{\mathbf{K}} \to \underline{\mathbf{S}}$ be a pre-ultrafunctor between the pre-ultracategories $\underline{\mathbf{K}}$ and $\underline{\mathbf{S}}$, and let δ be an ultramorphism in $\mathbf{K}$ and δ' an ultramorphism in $\mathbf{S}$, both of type $\langle \Gamma, k, \ell\rangle$.

Recall that in the terminology of the definition 3.3.11, δ is a natural transformation $(k) \xrightarrow{\delta} (\ell)$ of the evaluation functors

$$(k), (\ell) : \mathrm{Hom}(\Gamma, \mathbf{K}) \to \mathbf{K}.$$

(Resp. δ', $\mathbf{S}$.)

One would like to be able to say that for any ultradiagram $\mathscr{M} \in \mathrm{Hom}(\Gamma, \mathbf{K})$, we can apply X to produce a "pushforward" ultradiagram $X \circ \mathscr{M}$ in $\mathrm{Hom}(\Gamma, \mathbf{S})$. However, since we defined ultradiagrams "strictly" (by requiring that there is a literal equality between $\mathscr{M}(\beta)$ for β a bound node with ultraproduct data $(I, \mathcal{U}, g)$ and $\prod_{i \to \mathcal{U}} \mathscr{M}(g(i)))$, this only happens if (X, Φ) is a strict preultrafunctor (c.f. 3.4.1). So, we will do the next best thing and "strictify" $X \circ \mathscr{M}$.

Definition 3.3.15. ([12], Section 3) Let $(X, \Phi) : \underline{\mathbf{K}} \to \underline{\mathbf{S}}$ be a pre-ultrafunctor between the pre-ultracategories $\underline{\mathbf{K}}$ and $\underline{\mathbf{S}}$. We define an ultradiagram $X\mathscr{M} : \Gamma \to \underline{\mathbf{S}}$ as follows:

1. If γ is a free node, then $X\mathscr{M}(\gamma) \stackrel{\text{df}}{=} X \circ \mathscr{M}(\gamma)$.
2. If β is a bound node with ultraproduct data $(I, \mathcal{U}, g)$, then $X\mathscr{M}(\beta) \stackrel{\text{df}}{=} \prod_{i \to \mathcal{U}} X \circ \mathscr{M}(g(i))$.

There is an obvious natural isomorphism of functors $\nu : X \circ \mathscr{M} \simeq X\mathscr{M}$ whose component ν_β at a bound node β with ultraproduct data $(I, \mathcal{U}, g)$ is the appropriate component of the transition isomorphism $\nu_\beta \stackrel{\text{df}}{=} \Phi_{X \circ \mathscr{M}(g(i))}$ and whose component at a free node γ is just the identity map $\nu_\gamma \stackrel{\text{df}}{=} \mathrm{id}_{X \circ \mathscr{M}(\gamma)}$.

Definition 3.3.16. ([12], Section 3) Let $(X, \Phi) : \underline{\mathbf{K}} \to \underline{\mathbf{S}}$ be a pre-ultrafunctor between the pre-ultracategories $\underline{\mathbf{K}}$ and $\underline{\mathbf{S}}$, and let δ be an ultramorphism in $\mathbf{K}$ and δ' an ultramorphism in $\mathbf{S}$, both of type $\langle \Gamma, k, \ell \rangle$.

We say that X **carries** δ **into** δ' (prototypically, δ and δ' will both be canonically defined in the same way in both $\mathbf{K}$ and $\mathbf{S}$ and in this case we say that δ has been preserved) if for every ultradiagram $\mathscr{M} \in \mathrm{Hom}(\Gamma, \mathbf{K})$, the diagram

$$\begin{tikzcd} X\left(\mathscr{M}(k)\right) \arrow[r, "X(\delta_{\mathscr{M}})"] \arrow[d, "\nu_k"'] & X\left(\mathscr{M}(\ell)\right) \arrow[d, "\nu_\ell"] \\ (X\mathscr{M})(k) \arrow[r, "\delta'_{X\mathscr{M}}"'] & (X\mathscr{M})(\ell) \end{tikzcd}$$

commutes.

3.3.4 The definitions of ultracategory and ultrafunctor

Denote the class of all ultramorphisms in **Set** by $\Delta(\mathbf{Set})$.

Definition 3.3.17. ([12], Section 3) An **ultracategory K** is a pre-ultracategory (c.f. 3.3.1) K whose ultramorphisms are "fibered over" those of **Set**: we additionally require a specification of an ultramorphism $\delta_{\mathbf{K}}$ associated with any $\delta \in \Delta(\mathbf{Set})$ such that $\delta_{\mathbf{K}}$ is of the same type (Γ, k, ℓ) as δ.

Definition 3.3.18. ([12], Section 3) We define an **ultrafunctor** $X : \mathbf{K} \to \mathbf{S}$ between ultracategories **K**, **S** to be a pre-ultrafunctor (c.f. 3.3.2) which respects the fibering over **Set**: for every $\delta \in \Delta(\mathbf{Set})$, X carries $\delta_{\mathbf{K}}$ into $\delta_{\mathbf{S}}$ (in the sense of the definition 3.3.16 above) for all $\delta \in \Delta(\mathbf{Set})$.

Definition 3.3.19. A map between ultrafunctors is just an ultratransformation 3.3.2 of the underlying pre-ultrafunctors.

We write $\mathbf{Ult}(\mathbf{Mod}(T), \mathbf{Set})$ for the category of ultrafunctors $\mathbf{Mod}(T) \to \mathbf{Set}$.

3.4 The ultracategory structure on $\mathbf{Mod}(T)$

$\mathbf{Mod}(T)$ is canonically equipped with the structure of a pre-ultracategory by "lifting" the canonical pre-ultracategory structure on **Set**: an ultraproduct of models is just the ultraproduct of the underlying sets of the models.

We now describe how to additionally canonically equip $\mathbf{Mod}(T)$ with the structure of an ultracategory ("lifting", as before, the canonical ultracategory structure on **Set**).

Lemma 3.4.1. *Let $\mathscr{M} : \Gamma \to \mathbf{Mod}(T)$ be an ultradiagram. Let $X : \mathbf{Mod}(T) \to \mathbf{Set}$ be a strict pre-ultrafunctor. Then:*

1. *$X \circ \mathscr{M} : \Gamma \to \mathbf{Set}$ is an ultradiagram.*

2. if $\eta : \mathscr{M}_1 \to \mathscr{M}_2$ is an ultradiagram morphism, then $X\eta \overset{\text{df}}{=} \{X(\eta_\gamma)\}_{\gamma \in \Gamma}$ is an ultradiagram morphism $X \circ \mathscr{M}_1 \to X \circ \mathscr{M}_2$.

Proof. If β is a bound node of Γ with ultraproduct data $(I, \mathcal{U}, g)$, then

1. $$X \circ \mathscr{M}(\beta) = X\left(\prod_{i \to \mathcal{U}} \mathscr{M}(g(i))\right) = \prod_{i \to \mathcal{U}} X \circ \mathscr{M}(g(i)),$$

   whence strictness of the pre-ultrafunctor X. Thus the ultradiagram condition 3.3.9 is satisfied.

2. $$X(\eta_\beta) = X\left(\prod_{i \to \mathcal{U}} \eta_{g(i)}\right) = \prod_{i \to \mathcal{U}} X\left(\eta_{g(i)}\right),$$

   whence strictness of the pre-ultrafunctor X. Thus the ultradiagram morphism condition 3.3.10 is satisfied.

□

Definition 3.4.2. ([12], Section 3) We make the pre-ultracategory $\mathbf{Mod}(T)$ into an ultracategory by specifying, for each ultramorphism δ in $\Delta(\mathbf{Set})$ of type (Γ, k, ℓ), for every ultradiagram $\mathscr{M} : \Gamma \to \mathbf{Mod}(T)$, and for every object $A \in \mathbf{Def}(T)$,

$$\left(\left(\delta_{\mathbf{Mod}(T)}\right)_{\mathscr{M}}\right)_A \overset{\text{df}}{=} \delta_{\mathrm{ev}_A \circ \mathscr{M}}.$$

Remembering that $\delta_{\mathbf{Mod}(T)}$ is supposed to be a natural transformation of evaluation functors on ultradiagrams, and elementary embeddings are natural transformations, the equation displayed above reads: the component at the definable set A of the component at $\mathscr{M}$ of the ultramorphism $\delta_{\mathscr{M}(T)}$ is defined to be the component at the ultradiagram $\mathrm{ev}_A \circ \mathscr{M}$ of δ.

It is easy to verify that $\delta_{\mathbf{Mod}(T)}$ so defined is an ultramorphism, using the previous lemma.

Proposition 3.4.3. *Let $A \in \mathbf{Def}(T)$. Then the strict pre-ultrafunctor $\mathrm{ev}_A : \mathbf{Mod}(T) \to \mathbf{Set}$ is an ultrafunctor.*

Proof. Setting up the preservation of ultramorphisms condition 3.3.16, it remains to check that the diagram

$$\begin{tikzcd} \mathrm{ev}_A(\mathscr{M}(k)) \arrow[r, "\mathrm{ev}_A\left(\left(\delta_{\mathbf{Mod}(T)}\right)_{\mathscr{M}}\right)"] \arrow[d, equal] & \mathrm{ev}_A(\mathscr{M}(\ell)) \arrow[d, equal] \\ \mathrm{ev}_A\left(\mathscr{M}(k)\right) \arrow[r, "\delta_{\mathrm{ev}_A \circ \mathscr{M}}"'] & \mathrm{ev}_A(\mathscr{M}(\ell)) \end{tikzcd}$$

commutes. So,

$$\mathrm{ev}_A\left(\left(\delta_{\mathbf{Mod}(T)}\right)_{\mathscr{M}}\right) = \left(\left(\delta_{\mathbf{Mod}(T)}\right)_{\mathscr{M}}\right)_A,$$

which was defined above to be $\delta_{\mathrm{ev}\,A \circ \mathscr{M}}$. □

3.5 Strong conceptual completeness

There is a canonical evaluation functor

$$\widetilde{\mathrm{ev}} : \mathbf{Def}(T) \to \mathbf{Ult}(\mathbf{Mod}(T), \mathbf{Set})$$

sending each definable set $A \in T$ to its corresponding ultrafunctor $\widetilde{\mathrm{ev}}_A$, and we now have the following picture of factorizations of the original evaluation map $\mathrm{ev} : \mathbf{Def}(T) \to [\mathbf{Mod}(T) \to \mathbf{Set}]$:

$$\begin{tikzcd} \mathbf{Def}(T) \arrow[r, "\widetilde{\mathrm{ev}}"] \arrow[dr, "\widehat{\mathrm{ev}}"'] \arrow[ddr, bend right, "\mathrm{ev}"'] & \mathbf{Ult}(\mathbf{Mod}(T), \mathbf{Set}) \arrow[d, hook] \\ & \mathbf{PUlt}(\mathbf{Mod}(T), \mathbf{Set}) \arrow[d] \\ & {[\mathbf{Mod}(T), \mathbf{Set}]} \end{tikzcd}$$

Now, we can state strong conceptual completeness.

Theorem 3.5.1. *([12], Section 4) $\widetilde{\mathrm{ev}} : \mathbf{Def}(T) \to \mathbf{Ult}(\mathbf{Mod}(T), \mathbf{Set})$ is an equivalence of categories.*

Chapter 4

Δ-functors and definability for $\aleph_0$-categorical theories

In this chapter, we apply (pre)-ultracategories and Δ-functors to deduce a definability criterion for $\aleph_0$-categorical theories (Theorem 4.3.2): a functor $X : \mathbf{Mod}(T) \to \mathbf{Set}$ is definable, i.e. isomorphic as a functor to $\mathrm{ev}_{\varphi(x)}$ for some $\varphi(x) \in T$, if and only if there is some transition isomorphism Φ such that (X, Φ) is a Δ-functor.

This shows that for $\aleph_0$-categorical theories, the rest of the ultramorphisms 3.3.11 that were part of Makkai's reconstruction data for strong conceptual completeness are unnecessary for checking definability.

The result 4.3.2 is related to, but distinct from, Makkai's strong conceptual completeness. From 4.3.2, we know that if (X, Φ) is a Δ-functor, then the underlying functor X is isomorphic to an evaluation functor. This situation does not necessarily imply that (X, Φ) is an ultrafunctor. A counterexample is given in 8.1, where a definable functor is expanded by a transition isomorphism to a non-ultrafunctor. As the counterexample shows, we need to exploit the $\aleph_0$-categoricity assumption further before we can deduce strong conceptual completeness for $\aleph_0$-categorical theories.

Indeed, later we will prove a coherence criterion (Theorem 7.2.1) for objects in the classifying toposes of first-order theories, specialize to $\aleph_0$-categorical T, and deduce as a corollary Theorem 7.2.2, which says that any Δ-functor $(X, \Phi) : \mathbf{Mod}(T) \to \mathbf{Set}$ *is* an ultrafunctor, completing our deduction of strong conceptual completeness for $\aleph_0$-categorical theories.

4.1 Δ-functors and the finite support property

Definition 4.1.1. We say a functor $X : \mathbf{Mod}(T) \to \mathbf{Set}$ has the *finite support property* (is fsp, has fsp) if for every $M \in \mathbf{Mod}(T)$, for every $x \in X(M)$, there exists an $a \in M$ such that for every pair of elementary embeddings $h_1, h_2 : M \to N$, $h_1(a) = h_2(a) \implies Xh_1(x) = Xh_2(x)$.

As a warm-up to the theorem 4.3.2, we will show in general that if $X : \mathbf{Mod}(T) \to \mathbf{Set}$ is a Δ-functor, X must map $\mathrm{Aut}(M)$ *continuously* to $\mathrm{Sym}(X(M))$.

Proposition 4.1.2. *Let T be any theory, and let $(X, \Phi) : \mathbf{Mod}(T) \to \mathbf{Set}$ be a Δ-functor. Then for any model $M \models T$, the restriction of X to a map* $\mathrm{Aut}(M) \to \mathrm{Sym}(X(M))$ *is a continuous group homomorphism (where both groups are topologized by pointwise convergence).*

Proof. Since X is a functor, its restriction to $\mathrm{Aut}(M)$ is a group homomorphism. To check continuity, let $\mathcal{D}$ be a directed partial order indexing a net of automorphisms $[\sigma_\alpha]_{\alpha \in \mathcal{D}}$. It suffices to check that if $[\sigma_\alpha]_{\alpha \in \mathcal{D}} \to \sigma$ in $\mathrm{Aut}(M)$, then $[X\sigma_\alpha]_{\alpha \in \mathcal{D}} \to X\sigma$ in $\mathrm{Sym}(X(M))$.

We will suppose not and take an ultraproduct of counterexamples. So suppose that $[X\sigma_\alpha]_{\alpha \in \mathcal{D}}$ does not converge to $X\sigma$. The basic open neighborhoods $B_{c \mapsto d}$ of $X\sigma$ are parametrized by tuples c, d of the same sort, and they look like this:

$$B_{c \mapsto d} \stackrel{\mathrm{df}}{=} \{\rho : X(M) \to X(M) \,|\, \rho(c) = d\}.$$

Since $[X\sigma_\alpha]_{\alpha\in\mathcal{D}}$ does not converge to $X\sigma$, then there exists some neighborhood $B_{c\mapsto d}$ such that for every $\alpha \in \mathcal{D}$, there exists an $\alpha' \geqslant \alpha \in \mathcal{D}$ such that $X\sigma_{\alpha'} \notin B_{c\mapsto d}$.

Now, let I be the underlying set of $\mathcal{D}$, and consider the collection of subsets $\{P_\alpha \subseteq I\}_{\alpha\in\mathcal{D}}$, where each P_α is the set of all $\beta \in \mathcal{D}$ such that $\beta \geqslant \alpha$. Since $\mathcal{D}$ was a directed partial order, $\{P_\alpha\}_{\alpha\in\mathcal{D}}$ has the finite intersection property, and can therefore be completed to an ultrafilter $\mathcal{U}$.

Then consider the ultraproduct of automorphisms

$$[X\sigma_{\alpha'}]_{\alpha\to\mathcal{U}} : X(M)^{\mathcal{U}} \to X(M)^{\mathcal{U}}.$$

Let $\Delta_{X(M)}$ be the diagonal embedding of $X(M)$ into $X(M)^{\mathcal{U}}$. Since every $X\sigma_{\alpha'}$ sends c to $d' \neq d$, $[X\sigma_{\alpha'}]_{\alpha\to\mathcal{U}}$ sends $\Delta_{X(M)}(c)$ to $\Delta_{X(M)}(d') \neq \Delta_{X(M)}(d)$. Therefore,

$$[X\sigma_{\alpha'}]_{\alpha\to\mathcal{U}} \circ \Delta_{X(M)} \neq [X\sigma]_{\alpha\to\mathcal{U}} \circ \Delta_{X(M)}.$$

By the definition 3.3.14 of a Δ-functor, we can replace $\Delta_{X(M)}$ with $\Phi_{(M)} \circ X(\Delta_M)$. By the definition 3.3.2 of a pre-ultrafunctor, we can replace $[X\sigma_{\alpha'}]_{\alpha\to\mathcal{U}}$ and $[X\sigma]_{\alpha\to\mathcal{U}}$ with

$$\Phi_{(M)} \circ X\left([\sigma_{\alpha'}]_{\alpha\to\mathcal{U}}\right) \circ \Phi^{-1}_{(M)} \text{ and } \Phi_{(M)} \circ X\left([\sigma]_{\alpha\to\mathcal{U}}\right) \circ \Phi^{-1}_{(M)}.$$

Substituting into the displayed inequality above and letting inverse transition isomorphisms cancel out, we obtain

$$\Phi_{(M)} \circ X\left([\sigma_{\alpha'}]_{\alpha\to\mathcal{U}}\right) \circ X(\Delta_M) \neq \Phi_{(M)} \circ X\left([\sigma]_{\alpha\to\mathcal{U}}\right) \circ X(\Delta_M)$$

and since $\Phi_{(M)}$ is a bijection, we may omit it:

$$X\left([\sigma_{\alpha'}]_{\alpha\to\mathcal{U}}\right) \circ X(\Delta_M) \neq X\left([\sigma]_{\alpha\to\mathcal{U}}\right) \circ X(\Delta_M).$$

Since X is a functor, we conclude that

$$X\left([\sigma_{\alpha'}]_{\alpha\to\mathcal{U}} \circ \Delta_M\right) \neq X\left([\sigma]_{\alpha\to\mathcal{U}} \circ \Delta_M\right)$$

and since X is certainly a function from $\mathbf{Mod}(T)(M, M^{\mathcal{U}}) \to \mathbf{Set}\left(X(M), X\left(M^{\mathcal{U}}\right)\right)$, this means that

$$[\sigma_{\alpha'}]_{\alpha \to \mathcal{U}} \circ \Delta_M \neq [\sigma]_{\alpha \to \mathcal{U}} \circ \Delta_M.$$

But this inequality says that there is some $a \in M$ such that for every α, there is an α' such that $\{\sigma_{\alpha'}(a)\}_\alpha$ disagrees with $\{\sigma(a)\}_\alpha$ on some $\mathcal{U}$-large set of indices P. Letting $c = a$ and $d = \sigma(c)$, we have that a $\mathcal{U}$-large subset of $\{\sigma_{\alpha'}(a)\}_\alpha$ lies outside of the basic open $B_{c \mapsto d} \ni \sigma$. Since $\mathcal{U}$ contains all the principal filters in $\mathcal{D}$, we have that for every $\alpha \in \mathcal{D}$, the intersection $P \cap P_\alpha$ is nonempty. So, for the basic open $B_{c \mapsto d} \ni \sigma$, we have that for every α we can find some $\alpha'' \in P \cap P_\alpha$ such that $\sigma_{\alpha''} \notin B_{c \mapsto d}$. Therefore, $[\sigma_\alpha]_{\alpha \in \mathcal{D}}$ does not converge to σ, which is the contrapositive. □

Since for any T and $M \models T$, $\mathrm{End}(M)$ is the closure of $\mathrm{Aut}(M)$ inside the product space M^M, one easily modifies the above proof to obtain:

Theorem 4.1.3. *Let T and T' be any two theories. If $(X, \Phi) : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ is a Δ-functor, then for each $M \in \mathbf{Mod}(T)$,*

$$X_M \stackrel{\text{df}}{=} X \restriction \mathrm{End}(M) : \mathrm{End}(M) \to \mathrm{End}(X(M))$$

is continuous.

Theorem 4.1.4. *([12], Section 4) Let $(X, \Phi) : \mathbf{Mod}(T) \to \mathbf{Set}$ be a Δ functor. Then X is fsp.*

Proof. Towards the contrapositive, suppose X is not fsp. Then there is some M and $x \in X(M)$ such that for every tuple $a \in M$, there exists elementary embeddings $h_a, h'_a : M \to N_a$ such that $h_a(a) = h'_a(a)$ while $Xh_a(x) \neq Xh'_a(x)$.

As in the ultraproduct proof 3.2.3 of compactness, let I index all the finite subsets (i.e. tuples) of M. Let $\mathcal{U}$ be an ultrafilter completing the collection $\{P_i\}_{i \in I}$ where P_i is the set of all $j \in I$ such that, viewed as finite subsets of M, $j \supseteq i$; this collection has the finite intersection property, so is contained in some ultrafilter.

Now, take the ultraproducts $\mathbf{h}$ and $\mathbf{h}'$ of h_a and h'_a. On any element $[a]_{i \to \mathcal{U}}$ of the diagonally embedded copy of M in $M^{\mathcal{U}}$, $\mathbf{h}$ and $\mathbf{h}'$ agree on $[a]$ whenever $b \supseteq a$. Hence, this happens on P_a, which was in $\mathcal{U}$.

Therefore, the maps $\mathbf{h}, \mathbf{h}' : M^{\mathcal{U}} \to \prod_{\mathcal{U}} N_a$ are equalized by $\Delta_M : M \hookrightarrow M^{\mathcal{U}}$.

By assumption, this is not preserved by the functor X, so X must have failed to preserve Δ_M or an ultraproduct. □

Remark 4.1.5. An fsp functor is not necessarily the underlying functor of a Δ-functor. For example, if p is a complete non-isolated type, then the functor $X : \mathbf{Mod}(T) \to \mathbf{Set}$ taking each model M to its realizations $p(M)$ of p is fsp (if there is a realization, then it is its own support inside the model).

However, this X does not commute with ultraproducts (with the obvious choice of transition map): if M omits p, then $X(M) = \varnothing$. The ultraproduct of an empty set is empty, but since $M^{\mathcal{U}}$ realizes p, X is not a Δ-functor.

Somewhat less trivially, if X is definable then the infinite disjoint union $\bigsqcup_{i \in I} X$ again has fsp (every point is its own support), but with the obvious choice of transition map is not definable.

Later, we will see that in general these two examples are "absolutely undefinable", in the sense that there is no isomorphism whatsoever to any definable functor.

Finally, we point out that 4.1.3 and 4.1.4 are really saying the same thing:

Theorem 4.1.6. $X : \mathbf{Mod}(T) \to \mathbf{Set}$ *is fsp if and only if it induces continuous maps on endomorphism monoids.*

Proof. Suppose X is fsp. Fix M. For any finite tuple $x \in X(M)$ with support a_x, we have from the definition 4.1.1 of fsp that whenever $\sigma a_x = \mathrm{id}_M a_x$, $X\sigma x = \mathrm{id}_{X(M)} x$. Therefore, $\mathrm{Stab}(a_x) \subseteq X^{-1}(\mathrm{Stab}(x))$, so $X^{-1}(\mathrm{Stab}(x))$ is open. Since x was an arbitrary finite tuple and the pointwise convergence topology has a basis of neighborhoods of the identity given by stabilizers of finite tuples, this means that X restricts to a continuous map between endomorphism monoids equipped with the topology of pointwise convergence.

On the other hand, suppose X induces continuous monoid maps at each M. Then for every finite tuple $x \in X(M)$, $X^{-1}(\mathrm{Stab}(x))$ is open, hence contains some basic open neighborhood of the identity of the form $\mathrm{Stab}(a_x)$, for some a_x which we put as the support of x. $\square$

4.2 Failure of $\overline{F}$ to preserve the ultracategory structure

In [4], a pair of $\aleph_0$-categorical structures $M \models T$ and $M' \models T'$ are constructed which have isomorphic endomorphism monoids $\mathrm{End}(M) \simeq \mathrm{End}(M')$ that are *not* isomorphic as topological monoids. By 2.9.5 and 2.3.20, T is not bi-interpretable with T'. With what we have so far, we can see the failure of bi-interpretability at the level of ultracategories.

By 2.10.1, we know that the isomorphism of endomorphism monoids $F : End(M) \simeq \mathrm{End}(M') : G$ induces an equivalence of categories of models $\overline{F} : \mathbf{Mod}(T) \simeq \mathbf{Mod}(T') : \overline{G}$. By strong conceptual completeness 3.5.1, if there were a way of expanding $\overline{F}$ and $\overline{G}$ to ultrafunctors $(\overline{F}, \Phi) : \mathbf{Mod}(T) \simeq \mathbf{Mod}(T') : (\overline{G}, \Psi)$, this would induce an equivalence of categories of ultrafunctors $\mathbf{Ult}(\mathbf{Mod}(T), \mathbf{Set}) \simeq \mathbf{Ult}(\mathbf{Mod}(T'), \mathbf{Set})$, and hence of pretoposes $\mathbf{Def}(T') \simeq \mathbf{Def}(T)$. Therefore,

Theorem 4.2.1. *$\overline{F}$ or $\overline{G}$ cannot be the underlying functor of an ultrafunctor.*

Proof. Suppose there existed Φ and Ψ such that $(\overline{F}, \Phi)$ and $(\overline{G}, \Psi)$ are ultrafunctors. Then $(\overline{F}, \Phi)$ and $(\overline{G}, \Psi)$ are Δ-functors. By 4.1.3, F and G are then continuous. Since they already invert each other, $\mathrm{End}(M)$ and $\mathrm{End}(M')$ are isomorphic as topological monoids, a contradiction. □

4.3 A definability criterion for $\aleph_0$-categorical theories

Lemma 4.3.1. *Let T be any theory, and let $X : \mathbf{Mod}(T) \to \mathbf{Set}$ be a Δ-functor. Then X preserves filtered colimits of models: for any model N, if N can be written as the filtered colimit $N \simeq \varinjlim M_i$, then $X(N) \simeq \varinjlim X(M_i)$.*

Proof. First, we'll show that being a Δ-functor implies that elementary embeddings are sent to injective functions:

Claim: Let $f : M \to N$ be an elementary embedding. Then $X(f) : X(M) \to X(N)$ is injective.

Proof of claim. By Scott's lemma (see e.g. [3] for a proof), there is an ultrapower $M^{\mathcal{U}}$ of M and an elementary map $g : N \to M^{\mathcal{U}}$ such that the diagram

$$\begin{tikzcd} M^{\mathcal{U}} & \\ M \arrow[u, "\Delta_M"] \arrow[r, "f"'] & N \arrow[ul, "g"'] \end{tikzcd}$$

commutes. Since X was assumed to be a Δ-functor, the diagram

$$\begin{tikzcd} X(M)^{\mathcal{U}} & X\left(M^{\mathcal{U}}\right) \arrow[l, "\Phi_{(M)}"'] & \\ & X(M) \arrow[ul, "\Delta_{X(M)}"] \arrow[u, "X(\Delta_M)"'] \arrow[r, "X(f)"'] & X(N) \arrow[ul, "X(g)"'] \end{tikzcd}$$

commutes. Since $\Delta_{X(M)} : X(M) \hookrightarrow X(M)^{\mathcal{U}}$ is injective and $\Phi_{(M)}$ is a transition isomorphism, $X(\Delta_M)$ is injective, and therefore the composite $X(g) \circ X(f)$ is injective. Therefore, $X(f)$ was injective. □

Claim: For any $N \models T$, the collection of maps $\{X(f) \,|\, f : M \to N,\ M \text{ countable}\}$ jointly surject onto $X(N)$.

Proof of claim. Since N is covered by copies of countable models, we do know that $\{f \,|\, f : M \to N,\ M \text{ countable}\}$ jointly covers N.

Let I index the elementary embeddings from (representatives of isomorphism classes of) all countable models to N. Let $\mathcal{U}$ be a non-principal ultrafilter on I which contains the sets $P_{\vec{n}} \overset{\text{df}}{=} \{i \in I \,|\, \operatorname{im}(f_i) \ni \vec{n}\}$, which has the finite intersection property by the downward Lowenheim-Skolem theorem.

Consider the map

$$\prod_{i \to \mathcal{U}} M_i \overset{[f_i]_{i\to\mathcal{U}}}{\longrightarrow} N^{\mathcal{U}}.$$

The diagonal copy of N in $N^{\mathcal{U}}$ is in the image of this map: if $[n]_{i\to\mathcal{U}} \in N^{\mathcal{U}}$, then $\{i \in I \,|\, \exists m_i \text{ s.t. } f_i(m_i) = n\}$ is in $\mathcal{U}$, so $[f_i]_{i\to\mathcal{U}}[m_i]_{i\to\mathcal{U}} = [n]_{i\to\mathcal{U}}$. Pulling back $\Delta_N(N)$ along $[f_i]_{i\to\mathcal{U}}$, we obtain a map η from N into $\prod_{i\to\mathcal{U}} M_i$ such that the diagram

$$\begin{tikzcd}
N^{\mathcal{U}} & \\
N \arrow[u, "\Delta_N"] \arrow[r, "\eta"'] & \prod_{i\to\mathcal{U}} M_i \arrow[ul, "{[f_i]_{i\to\mathcal{U}}}"']
\end{tikzcd}$$

commutes.

Now apply X, obtaining the commutative diagram (it is easy to check that the extra subdiagrams involving $X(\eta)$ commute by $\Phi_{(N)}$ and $\Phi_{(M_i)}$ being isomorphisms):

$$\begin{tikzcd}
& X(N) \arrow[dl, "X(\Delta_N)"'] \arrow[dr, "\Delta_{X(N)}"] \arrow[ddl, "X(\eta)"] & \\
X\left(N^{\mathcal{U}}\right) \arrow[rr, "\Phi_{(N)}"] & & X(N)^{\mathcal{U}} \\
X\left(\prod_{i\to\mathcal{U}} M_i\right) \arrow[u, "{X([f_i]_{i\to\mathcal{U}})}"] \arrow[rr, "\Phi_{(M_i)}"'] & & X(M)^{\mathcal{U}}. \arrow[u, "{[X(f_i)]_{i\to\mathcal{U}}}"']
\end{tikzcd}$$

In particular,

$$\Delta_{X(N)} = [X(f_i)]_{i \to \mathcal{U}} \circ \Phi_{(M_i)} \circ X(\eta).$$

This implies that $\Delta_{X(N)}$ is contained inside the image of $[X(f_i)]_{i \to \mathcal{U}}$.

Now, suppose that the $X(f_i)$ did not cover $X(N)$. That is, suppose that there exists an $x \in X(N)$ such that x lies outside of the image of $X(f_i)$ for every $i \in I$. Then for any $[m_i]_{i \to \mathcal{U}} \in \prod_{i \to \mathcal{U}} M_i$, $f_i(m_i) \neq x$ for all $i \in I$. Therefore, $\Delta_{X(N)}(x)$ is not contained in the image of $[X(f_i)]_{i \to \mathcal{U}}$, a contradiction.

We conclude that $\{X(f) \mid f : M \to N\}$ jointly surjects onto $X(N)$. □

Claim: Present N as a filtered colimit of its countable submodels M_i. Then $X(N) \simeq \varinjlim X(M_i)$.

Proof of claim. Our two previous claims show that we may view $X(N)$ as the union of the $X(M_i)$'s. $\varinjlim X(M_i)$ can be canonically written as

$$\left(\bigsqcup_{i \in I} X(M_i)\right) / E$$

where $(x \in X(M_i)) \sim_E (y \in X(M_j))$ if and only if x and y become the same element in some $X(M_k)$ for M_k amalgamating M_i and M_j. It is easy to check that sending an $x \in X(N)$ to the E-class of an arbitrary lift $x' \in X(M_i)$ (for a choice of some $X(M_i)$ containing x') gives a bijection

$$X(N) \simeq \varinjlim X(M_i) \text{ by } x \mapsto [x']_E,$$

compatible over the $X(M_i)$'s. □

So far, we have shown that X preserves filtered colimits of countable models. But every model is a filtered colimit of countable models. Explicitly, if we have $N = \varinjlim_i N_i$ where the N_i are possible uncountable, we have that each $N_i = \varinjlim_j N_j^i$, so that we have written N as a filtered colimit of countable models N_j^i:

$$N = \varinjlim_i \varinjlim_j N_j^u = \varinjlim_{(i,j)} N_j^i$$

Then

$$X(N) \simeq \varinjlim_{(i,j)} X(N^i_j) \simeq \varinjlim_i \varinjlim_j X(N^i_j) \simeq \varinjlim_i X(N_i).$$

□

Theorem 4.3.2. *Let T be $\aleph_0$-categorical. A functor $X : \mathbf{Mod}(T) \to \mathbf{Set}$ is definable if and only if there is a transition isomorphism Φ such that (X, Φ) is a Δ-functor.*

Proof. If X is definable, then its isomorphism to an evaluation functor φ pulls back φ's transition isomorphism Φ' to a transition isomorphism Φ for X, and since (φ, Φ') was an ultrafunctor (X, Φ) is also (these are diagrammatic conditions on Φ' and so are invariant under conjugation by isomorphisms).

On the other hand, suppose that (X, Φ) is a Δ-functor. $\mathrm{Aut}(M)$ acts via X on $X(M)$, and so $X(M)$ splits up into $\mathrm{Aut}(M)$-orbits. For each representative x of these orbits, we know from the remarks following 4.1.2 that there is a tuple $a_x \in M$ which supports x, and the map $a_x \mapsto x$ induces an $\mathrm{Aut}(M)$-equivariant map from the orbit (type) of a_x to the orbit of x.

Therefore, each $\mathrm{Aut}(M)$-orbit of $X(M)$ is a quotient of an $\mathrm{Aut}(M)$-orbit of M by some $\mathrm{Aut}(M)$-invariant equivalence relation. Since M is $\aleph_0$-categorical, these equivalence relations are definable and all types are isolated by formulas, so we can write:

$$X(M) \simeq \bigvee_{i \in I} M(\varphi_i(x_i)) \simeq \bigsqcup_{i \in I} M(\varphi_i(x_i)).$$

By the previous lemma 4.3.1 and the fact that colimits always commute with colimits and definable functors always commute with filtered colimits of models, we conclude (writing $N = \varinjlim_j M_j$):

$$X(N) \simeq \varinjlim_j \left(\bigsqcup_{i \in I} \varphi_i(M_j) \right) \tag{4.1}$$

$$\simeq \bigsqcup_{i \in I} \left(\varinjlim_j \varphi_i(M_j) \right) \tag{4.2}$$

$$\simeq \bigsqcup_{i \in I} \left(\varphi_i(\varinjlim_j M_j) \right) \tag{4.3}$$

$$\simeq \bigsqcup_{i \in I} \varphi_i(N). \tag{4.4}$$

Now we will show that the I indexing the φ_i must be finite.

In the pre-ultrafunctor condition

$$\begin{tikzcd} X\left(\prod_{\mathcal{U}} M_i\right) \arrow[r, "\Phi_{\mathcal{U},(M_i)}"] \arrow[d, "X\left(\prod_{\mathcal{U}} f_i\right)"'] & \prod_{\mathcal{U}}\left(X(M_i)\right) \arrow[d, "\prod_{\mathcal{U}} X(f_i)"] \\ X\left(\prod_{\mathcal{U}} N_i\right) \arrow[r, "\Phi_{\mathcal{U},(N_i)}"'] & \prod_{\mathcal{U}}\left(X(N_i)\right), \end{tikzcd}$$

restricting our attention to just ultraproducts of automorphisms tells us that $\Phi_{(M_i)} : X\left(\prod_{i \to \mathcal{U}}\right) M_i \to \prod_{i \to \mathcal{U}} X(M_i)$ is a $\prod_{i \to \mathcal{U}} \mathrm{Aut}(M_i)$-equivariant bijection, and therefore induces a bijection on the orbits of the action on either side.

Let $\mathcal{U}$ be some ultrafilter such that $|I^{\mathcal{U}}| > |I|$. Then, at the countable model M, we have the bijection:

$$X\left(M^{\mathcal{U}}\right) \overset{\Phi_{(M)}}{\simeq} \left(X(M)\right)^{\mathcal{U}}.$$

Now, the left hand side is $\bigsqcup_{i \in I} \varphi_i\left(M^{\mathcal{U}}\right)$. Each $\varphi_i\left(M^{\mathcal{U}}\right)$ is actually an $\mathrm{Aut}(M)^{\mathcal{U}}$-orbit, since $\varphi_i(M)$ was an $\mathrm{Aut}(M)$-orbit. Therefore, the number of $\mathrm{Aut}(M)^{\mathcal{U}}$-orbits on the left hand side is $|I|$.

On the right hand side, we have $\left(\bigsqcup_{i \in I} \varphi_i(M)\right)^{\mathcal{U}}$. Two points $[x_i]_{i \to \mathcal{U}}$ and $[y_i]_{i \to \mathcal{U}}$ are $\mathrm{Aut}(M)^{\mathcal{U}}$-conjugate if and only if there exists a $P \in \mathcal{U}$ such that for all $j \in P$,

$\varphi_{x_j} = \varphi_{y_j}$ (where φ_{x_i} means which φ_k x_i came from.) But, this is the same as saying $[\varphi_{x_j}]_{j\to\mathcal{U}} = [\varphi_{y_j}]_{j\to\mathcal{U}}$. So the number of orbits on the right hand side is $|I|^{\mathcal{U}}$.

Therefore, $|I^{\mathcal{U}}| = |I|$, so I must be finite. Hence there is a formula $\varphi(x)$ such that $X(N) \simeq \varphi(N)$ for all $N \models T$. Since for each N, this isomorphism $X(N) \simeq \varphi(N)$ is induced via filtered colimits by $X(M) \simeq \varphi(M)$, this is a natural isomorphism, so X is definable. $\square$

4.3.1 $\mathrm{Aut}(M)^{\mathcal{U}}$ orbit-counting

Besides the observation 4.1.3 that Δ-functors induce continuous maps of automorphism groups, the key step in the proof of the theorem 4.3.2 was counting $\mathrm{Aut}(M)^{\mathcal{U}}$-orbits in an ultrapower, coming from the fact that pre-ultrafunctors $X : \mathbf{Mod}(T) \to \mathbf{Set}$ are defined by requiring all squares

$$\begin{tikzcd}
X\left(\prod_{\mathcal{U}} M_i\right) \arrow[r, "\Phi_{\mathcal{U},(M_i)}"] \arrow[d, "X\left(\prod_{\mathcal{U}} f_i\right)"'] & \prod_{\mathcal{U}}\left(X(M_i)\right) \arrow[d, "\prod_{\mathcal{U}} X(f_i)"] \\
X\left(\prod_{\mathcal{U}} N_i\right) \arrow[r, "\Phi_{\mathcal{U},(N_i)}"'] & \prod_{\mathcal{U}}\left(X(N_i)\right).
\end{tikzcd}$$

to commute; in particular, when $N_i = M_i$ for all i, this says that X is necessarily $\mathrm{Aut}(M)^{\mathcal{U}}$-equivariant, where

$$\mathrm{Aut}(M)^{\mathcal{U}} \overset{\text{df}}{=} \{[\sigma_i]_{i\to\mathcal{U}} \mid \sigma_i \in \mathrm{Aut}(M)\},$$

where $[\sigma_i]_{i\to\mathcal{U}} : M^{\mathcal{U}} \to M^{\mathcal{U}}$ is defined by pointwise application on elements $[x_i]_{i\to\mathcal{U}}$ of the ultrapower.

Note that while $M^{\mathcal{U}}$ might be saturated and so $p(M^{\mathcal{U}})$ is transitively acted upon by the full automorphism group $\mathrm{Aut}(M^{\mathcal{U}})$, this is not true under the $\mathrm{Aut}(M)^{\mathcal{U}}$-action: a non-isolated type p will have realizations $[x_i]_{i\to\mathcal{U}}$ which are $\mathcal{U}$-often *not* realizations of p.

For example, take a countable model of the (theory of the infinite set + countably many distinct constants), such that the non-isolated type "I am not any of the constants" is realized by some a. In a countable ultrapower of this model, $\Delta(a)$ is not $\mathrm{Aut}(M)^{\mathcal{U}}$-conjugate to $[c_i \mid i \in \omega]_{i \to \mathcal{U}}$ (in fact, since this element of $M^{\mathcal{U}}$ comes from a sequence of constants, this is a fixed point of the $\mathrm{Aut}(M)^{\mathcal{U}}$-action.)

We can write down an explicit description of the $\mathrm{Aut}(M)^{\mathcal{U}}$-orbits of $p(M^{\mathcal{U}})$ for a complete type p.

Lemma 4.3.3. *Let p be a complete type of T. Let $\left(a_i \in M_i \,\middle|\, M_i \models T\right)$ be a sequence of elements in possibly distinct models. Let $\mathcal{U}$ be a non-principal ultrafilter on I.*

Then $\mathrm{tp}([a_i]_{i \to \mathcal{U}}) = p$ if and only if in the Stone space, the sequence $\left(p_i \overset{\text{df}}{=} \mathrm{tp}(a_i)\right)_{i \in I}$ $\mathcal{U}$-converges to p.

Proof. Suppose that $[a_i]_{i \to \mathcal{U}} \models p$. Then for each $\varphi \in p$, $\mathcal{U}$-often, $a_i \in \varphi(M_i)$. Hence for each D_φ the basic open neighborhood of p corresponding to φ in the Stone space, $\mathcal{U}$-often, $p_i \in D_\varphi$. Hence $p_i \xrightarrow{\mathcal{U}} p$.

Now suppose that $p_i \xrightarrow{U} p$. Let $\varphi \in p$. Then $\mathcal{U}$-often, $p_i \in D_\varphi$, equivalently, $\mathcal{U}$-often, $a_i \in \varphi(M_i)$. Hence $[a_i]_{i \to \mathcal{U}} \models p$. □

Theorem 4.3.4. *The $\prod_{i \to \mathcal{U}} \mathrm{Aut}(M_i)$-orbits of $p\left(\prod_{i \to \mathcal{U}} M_i\right)$ refine the equivalence classes $[p_i]_{i \to \mathcal{U}}$ where each p_i is realized in M_i and $p_i \xrightarrow{\mathcal{U}} p$ in the Stone space.*

Furthermore, if the M_i are homogeneous (so that any two realizations of the same type in each M_i are $\mathrm{Aut}(M_i)$-conjugate), then we can improve "refine" to "are exactly".

Proof. It suffices to show that the map

$$\mathrm{tp}^{\mathcal{U}} \overset{\text{df}}{=} \left([a_i]_{i \to \mathcal{U}} \mapsto [\mathrm{tp}(a_i)]_{i \to \mathcal{U}}\right) : p\left(\prod_{i \to \mathcal{U}} M_i\right) \to \mathrm{St}(T)^{\mathcal{U}}$$

is constant on each $\mathrm{Aut}(M)^{\mathcal{U}}$ orbit.[1]

[1] Note that this only means that we have a well-defined surjection from $\mathrm{Aut}(M)^{\mathcal{U}}$-orbits onto the

Let $[a_i]_{i\to\mathcal{U}}$ be $\mathrm{Aut}(M)^{\mathcal{U}}$-conjugate to $[b_i]_{i\to\mathcal{U}}$. Then (ultrafilter-often), $\mathrm{tp}(a_i) = \mathrm{tp}(b_i)$ so $\mathrm{tp}^{\mathcal{U}}\left([a_i]_{i\to\mathcal{U}}\right) = \mathrm{tp}^{\mathcal{U}}\left([b_i]_{i\to\mathcal{U}}\right)$.

Now, suppose furthermore that in each M_i, any two realizations of the same type in M_i are $\mathrm{Aut}(M_i)$-conjugate. Then if two realizations $[a_i]_{i\to\mathcal{U}}$ and $[b_i]_{i\to\mathcal{U}}$ of p in $\prod_{i\to\mathcal{U}} M_i$ are *not* $\mathrm{Aut}(M_i)_{i\to\mathcal{U}}$-conjugate, it follows that $\mathcal{U}$-often, $\mathrm{tp}(a_i) \neq \mathrm{tp}(b_i)$.

Therefore, $\mathrm{tp}^{\mathcal{U}}([a_i]_{i\to\mathcal{U}}) \neq \mathrm{tp}^{\mathcal{U}}([b_i]_{i\to\mathcal{U}})$. □

Remark 4.3.5. With this theorem, the role of ω-categoricity in the orbit-counting argument for the proof of 4.3.2 is clear: there are only finitely many types in every sort, and all the types are isolated, so in the Stone space, the only sequences which approach these types are constant sequences of these types.

Therefore, counting $\mathrm{Aut}(M)^{\mathcal{U}}$-orbits of $\bigvee_{i\in I} p_i\left(M^{\mathcal{U}}\right)$ yields $|I|$, whereas counting $\mathrm{Aut}(M)^{\mathcal{U}}$-orbits of $\left(\bigvee_{i\in I} p_i(M)\right)^{\mathcal{U}}$ yields $|I^{\mathcal{U}}|$.

ultraproducts of sequences of types converging to p. To get injectivity also, we would need to show that the values are different for different orbits.

Chapter 5

Strictifications of pre-ultrafunctors

In this chapter, we prove a purely formal theorem comparing non-strict pre-ultrafunctors $\mathbf{Mod}(T) \to \mathbf{Set}$ (i.e. those whose transition isomorphisms are not all the identity map) with strict ones (i.e. those whose transition isomorphisms are all the identity map), showing how for any non-strict pre-ultrafunctor we may obtain an isomorphic strict pre-ultrafunctor. To carry out this construction we perform a transfinite induction on the "ultraproduct complexity" of models; this complexity is given in terms of an ordinal-valued rank.

5.1 Strict vs non-strict pre-ultrafunctors

Throughout, we will work with the usual (pre)ultracategory structures on $\mathbf{Mod}(T)$ and $\mathbf{Set}$. In particular, in Definition 3.1.2, we fixed once and for all the ultraproduct functors $[U] : \mathbf{Set}^I \to \mathbf{Set}$, and whenever we talk about ultraproducts of sets, we understand that we are applying those specific ultraproduct functors. As we saw in Definition 3.1.3, once the pre-ultracategory structure on $\mathbf{Set}$ has been fixed, this induces a "standard" pre-ultracategory structure on $\mathbf{Mod}(T)$, and so determines what the ultraproduct functors are for $\mathbf{Mod}(T)$, too.

In general, pre-ultrafunctors $\mathbf{Mod}(T) \to \mathbf{Set}$ are only required to "preserve ultraproducts up to a specified transition isomorphism", which means that the squares

$$\begin{tikzcd} X\left(\prod_{i \to \mathcal{U}} M_i\right) \arrow[r, "\Phi_{(M_i)}"] \arrow[d, "X\left(\prod_{i\to\mathcal{U}} f_i\right)"'] & \prod_{i \to \mathcal{U}} X\left(M_i\right) \arrow[d, "\prod_{i\to\mathcal{U}} X(f_i)"] \\ X\left(\prod_{i \to \mathcal{U}} N_i\right) \arrow[r, "\Phi_{(<_i)}"'] & \prod_{i \to \mathcal{U}} X\left(N_i\right) \end{tikzcd}$$

(ranging over all indexing sets I, ultrafilters $\mathcal{U}$ on I, and I-indexed sequences of elementary embeddings $(M_i \xrightarrow{f_i} N_i))$ commute, where $\Phi_{(M_i)}$ does not necessarily have to be the identity map, only some isomorphism.

Definition 5.1.1. If a pre-ultrafunctor X *does* have identity maps for all of it transition isomorphisms, we say that X is **strict**.

Remark 5.1.2. The proof of the Łos theorem 3.2.1 shows that the evaluation functors $M \mapsto M(X)$ for any (eq)-definable set X is a strict pre-ultrafunctor.

Remark 5.1.3. One might worry about being able to achieve strictness in the situation where there is a way to write a model N as two different ultraproducts, say $N = M^{\mathcal{U}}$ and $N = \prod_{i \to \mathcal{U}} M_i$; then, after applying a general pre-ultrafunctor X, we have

$$\begin{tikzcd} X\left(M^{\mathcal{U}}\right) \arrow[r, equal] \arrow[d, "\Phi_{(M)}"'] & X\left(\prod_{i \to \mathcal{U}} M_i\right) \arrow[d, "\Phi_{(M_i)}"] \\ X(M)^{\mathcal{U}} \arrow[r, "\sim"] & \prod_{i \to \mathcal{U}} X(M_i) \end{tikzcd}$$

where the bottom isomorphism is the unique map given by composing the isomorphisms in the rest of the diagram.

If we try to make X strict, say by setting $\Phi_{(M_i)}$ to be the identity, then we see that the other transition isomorphism $\Phi_{(M)}$ can't be the identity, which would present an obstruction to finding any strict pre-ultrafunctor on $\mathbf{Mod}(T)$.

However, as long as we are careful about what sets we assign to be ultraproducts (as when we made the conventions in Definition 3.1.2 about what the ultraproduct functors on $\mathbf{Set}$ precisely were), this situation never arises; we explain below.

Lemma 5.1.4. *If we have unidentical data* $(I,\mathcal{U},(M_i)_{i\in I}) \neq (I',\mathcal{U}',(M_i')_{i\in I'})$*, then* $\prod_{i\to\mathcal{U}} M_i \neq \prod_{i\to\mathcal{U}'} M_i'$.

Proof. Recalling our conventions made in Definition 3.1.2 about how we construct products and quotients in **Set**, we look at the three cases:

1. If $I \neq I'$, then for every definable set A,
$$\prod_{i\to I} M_i(A) = \left\{\{(a,i) \,\middle|\, a \in M_i(A)\}_{i\in I}\right\} \neq \left\{\{(a',i') \,\middle|\, a' \in M_{i'}'(A)\}_{i'\in I'}\right\} = \prod_{i'\in I'} M_{i'},$$
and so their quotients, which we think of as collections of equivalence classes, cannot literally be the same set.

2. If $\mathcal{U} \neq \mathcal{U}'$, then even if all the other data were the same, the quotients of the I-indexed products by $\mathcal{U}$ and $\mathcal{U}'$, which we think of as collections of equivalence classes, cannot be the same since $\mathcal{U}$ and $\mathcal{U}'$ are distinct.

3. Even if $I = I'$ and $\mathcal{U} = \mathcal{U}'$, if for some $i \in I$, we have distinct models $M_i \neq M_i'$, then by definition, for some definable set A, the sets $M_i(A)$ and $M_i'(A)$ are distinct. Then the I-indexed products are distinct and so are the quotients by $\mathcal{U}$.

□

Strong conceptual completeness [12] tells us that ultrafunctors $\mathbf{Mod}(T) \to \mathbf{Set}$ are definable, i.e. are isomorphic to evaluation functors of the kind in the previous paragraph. In particular, every ultrafunctor is isomorphic to a strict ultrafunctor, and we can think of the condition of preserving all ultramorphisms as forcing any non-identity transition isomorphisms of a given pre-ultrafunctor to be, in some way, "canonical".

The purpose of this section is to show that any non-strict pre-ultrafunctor is isomorphic to a strict one, and that the construction of this isomorphism respects the preservation of ultramorphisms. In particular, any non-strict Δ-functor is isomorphic to a strict Δ-functor and this gives another proof that any non-strict ultrafunctor is isomorphic to a strict ultrafunctor.

5.2 The ultraproduct rank of a model

Now we introduce the *ultraproduct rank* of a model, which will be an inductively-defined ordinal rank that measures how complicated it is to write a model up to isomorphism as a non-trivial ultraproduct of smaller models.

To avoid quantifying over proper classes of isomorphism types of models, we make an auxiliary definition:

Definition 5.2.1. Let κ be a regular cardinal. A **κ-bounded model** of T is a model $M' : \mathbf{Def}(T) \to \mathbf{Set}$ which factors through the full subcategory $\mathbf{Set}_\kappa$ of $\mathbf{Set}$ spanned by the hereditarily κ-small sets. (Equivalently, for any $A \in T$, $M'(A)$ must be a hereditarily κ-small set.

Since our theories have only a small number of definable sets, for any model M : $\mathbf{Def}(T) \to Set$, there exists some isomorphic κ-bounded model $M' : \mathbf{Def}(T) \to \mathbf{Set}_\kappa$.

Definition 5.2.2. The **ultraproduct rank** of a model M is an ordinal $\operatorname{upc}(M)$ which we define inductively as follows:

(i) If M is not isomorphic to a non-trivial ultraproduct of $|M|^+$-bounded models, then put $\operatorname{upc}(M) \overset{\text{df}}{=} 0$.

(ii) Otherwise, put

$$\operatorname{upc}(M) \overset{\text{df}}{=} \inf_{\left\{(M_j) \,\middle|\, \prod_{j \to \mathcal{U}} M_j \simeq M\right\}} \sup_j \operatorname{upc}(M_j)$$

(here the infimum runs over all sequences of $|M|^+$-bounded models $(M_j)_{j \in J}$ such that $\prod_{j \to \mathcal{U}} M_j \simeq M$ for some non-principal ultrafilter $\mathcal{U}$ on J.)

Remark 5.2.3. In part **(ii)** of the previous definition, we can derive from the cardinality of M a bound on the cardinality of the possible indexing sets J, and there are only a set's worth of κ-bounded models of T, so for each M we are only quantifying over a set's worth of things: the ultraproduct rank is well-defined.

To perform the construction in the next section, we will make some arbitrary choices. In particular, we will need to choose witnesses for the value of the ultraproduct rank.

Definition 5.2.4. If $\mathrm{upc}(M) = \alpha$, we define a **witness** for this to be a sequence $(M_j)_{j \in J}$ and an ultrafilter $\mathcal{U}$ on J such that $\prod_{h \to \mathcal{U}} M_j \simeq M$ and $\sup_j \mathrm{upc}(M_j) = \alpha$. Since the ordinals are well-ordered, witnesses always exist.

5.3 Constructing the isomorphism

Theorem 5.3.1. *For every non-strict pre-ultrafunctor $X : \mathbf{Mod}(T) \to \mathbf{Set}$, there exists a strict pre-ultrafunctor $X' : \mathbf{Mod}(T) \to \mathbf{Set}$ and an isomorphism $X \simeq X'$.*

Proof. We start building X' by asking that if M_1 and M_2 are rank 0, then

$$X' \left(M_1 \xrightarrow{f} M_2 \right) \overset{\mathrm{df}}{=} X(M_1) \xrightarrow{X(f)} X(M_2).$$

This defines X' on the full subcategory of rank 0 models and completes the base of the induction.

Now the induction step. If X' has already been defined on the full subcategory $\mathbf{C} \subseteq \mathbf{Mod}(T)$ of rank $< \alpha$ models, then fix choices of witnesses for anything in $\mathbf{C}$ extending any choices of witnesses we have made at an earlier stage, and extend X' to a full subcategory $\mathbf{C}' \subseteq \mathbf{Mod}(T)$ made of anything that is an ultraproduct of objects of $\mathbf{C}$ by setting $X'(M_1 \xrightarrow{f} M_2)$ to the dashed map below:

$$\left.\begin{array}{c} X'\left(\prod_{i \to \mathcal{U}} M_i\right) \\ {\scriptstyle X'(f)}\big\downarrow \\ X'\left(\prod_{j \to \mathcal{V}} N_j\right) \end{array}\right\} \stackrel{\text{df}}{=} \begin{tikzcd} \prod_{i \to \mathcal{U}} X'(M_i) \arrow[d, dashed, "!"'] & X\left(\prod_{i \to \mathcal{U}} M_i\right) \arrow[l, "\Phi'_{(M_i)}"'] \arrow[d, "X(f)"] \\ \prod_{j \to \mathcal{V}} X'(N_j) & X\left(\prod_{j \to \mathcal{V}} N_j\right) \arrow[l, "\Phi'_{(N_j)}"] \end{tikzcd},$$

where:

(i) $\Phi'_{(M_i)}$ (resp. (N_j)) is defined by the composition

$$\begin{tikzcd} X\left(\prod_{i \to \mathcal{U}} M_i\right) \arrow[r, dashed, "!"] \arrow[d, "\Phi_{(M_i)}"'] & \prod_{i \to \mathcal{U}} X'(M_i) \\ \prod_{i \to \mathcal{U}} X(M_i) \arrow[ur, "\prod_{i \to \mathcal{U}} \sigma_i"'] & \end{tikzcd},$$

and where:

(ii)

$$\sigma_i \stackrel{\text{df}}{=} \begin{cases} \mathrm{id}_{X(M_i)} & \text{if } \operatorname{upc}(M_i) = 0, \\ \Phi'_{(M^i_j)} & \text{if } X'(M_i) \text{ was defined at an earlier stage, and } \prod_{j \to \mathcal{W}} M^i_j = M_i \text{ is a witness for the nonzero ultraproduct rank of } M_i. \end{cases}$$

To complete the induction step, we have to extend X' to *all* the rank-α models (note that N being a rank-α model only means that N is *isomorphic* to an ultraproduct of rank $< \alpha$ models, not necessarily that N *is* an ultraproduct of rank $< \alpha$ models), so we choose a witness $\prod_{i \to \mathcal{U}} M_i \simeq N$ that $\operatorname{upc}(N) = \alpha$ In particular, we have chosen an isomorphism $\prod_{i \to \mathcal{I}} M_i \simeq N$. We then extend X' to the full subcategory of $\mathbf{Mod}(T)$ spanned by $\mathbf{C}' \cup \{N\}$ by decreeing that $X'(N) = X'\left(\prod_{i \to \mathcal{U}} M_i\right)$, and that the chosen isomorphism is sent to the identity. Doing this for all N, we extend X' to the full subcategory of rank-α-models.

Since we observed (Remark 5.2.3) that the ultraproduct rank is well-defined, every model is reached at some (possibly transfinite) stage of this construction.

X' is a functor because conjugating by Φ''s cancels out.

To check pre-ultrafunctoriality, we need the diagrams

$$\begin{tikzcd} X'\left(\prod_{i\to\mathcal{U}} M_i\right) \arrow[r, equal] \arrow[d, "X'\left(\prod_{i\to\mathcal{U}} f_i\right)"'] & \prod_{i\to\mathcal{U}} X'(M_i) \arrow[d, "\prod_{i\to\mathcal{U}} X'(f_i)"] \\ X'\left(\prod_{i\to\mathcal{U}} N_i\right) \arrow[r, equal] & \prod_{i\to\mathcal{U}} X'(N_i) \end{tikzcd}$$

to commute, i.e. that $X'\left(\prod_{i\to\mathcal{U}} f_i\right) = \prod_{i\to\mathcal{U}} X'(f_i)$.

So, unravelling the definitions, we calculate:

$$\begin{aligned} X'\left(\prod_{i\to\mathcal{U}} f_i\right) &\stackrel{?}{=} \prod_{i\to\mathcal{U}} X'(f_i) \\ \iff \Phi'_{(N_i)} \circ X\left(\prod_{i\to\mathcal{U}} f_i\right) \circ \left(\Phi'_{(M_i)}\right)^{-1} &\stackrel{?}{=} \prod_{i\to\mathcal{U}} X'(f_i) \\ \iff \prod_{i\to\mathcal{U}} \sigma_i^N \circ \Phi_{(N_i)} \circ X\left(\prod_{i\to\mathcal{U}} f_i\right) \circ \left(\Phi_{(M_i)}\right)^{-1} \circ \left(\prod_{i\to\mathcal{U}} \sigma_i^M\right)^{-1} &\stackrel{?}{=} \prod_{i\to\mathcal{U}} X'(f_i) \\ \iff \prod_{i\to\mathcal{U}} \sigma_i^N \circ \prod_{i\to\mathcal{U}} X(f_i) \circ \left(\prod_{i\to\mathcal{U}} \sigma_i^M\right)^{-1} &\stackrel{?}{=} \prod_{i\to\mathcal{U}} X'(f_i) \\ \iff \prod_{i\to\mathcal{U}} \left(\sigma_i^N X(f_i) \left(\sigma_i^M\right)^{-1}\right) &\stackrel{?}{=} \prod_{i\to\mathcal{U}} X'(f_i) \\ &\stackrel{?}{=} \prod_{i\to\mathcal{U}} \Phi'_{N_i} X(f_i) \left(\Phi'_{M_i}\right)^{-1} \\ &= \prod_{i\to\mathcal{U}} \sigma_i^N X(f_i) \left(\sigma_i^M\right)^{-1}. \end{aligned}$$

In the final step, we are observing that since Φ'_{N_i} is id_{N_i} if N_i was from the base case, Φ_{N_i} *is* σ_i^N (resp. M_i). □

5.4 Showing the constructed isomorphism respects preservation of ultramorphisms

Now we will show that the property of a pre-ultrafunctor (X, Φ) preserving an ultramorphism is invariant under the construction 5.3.1 of X'.

Heuristically, this should be true because preserving ultramorphisms is a "local", component-by-component property of the transition isomorphisms Φ, and the only source of the new data Φ' in the construction of X' was taking ultraproducts of components of the old Φ, with maybe some identity maps interspersed.

Rigorously, this follows from some definition-unraveling. To give an idea for it, we will first prove the special case that the preservation of the diagonal maps Δ is invariant under the construction of X'.

Proposition 5.4.1. *A pre-ultrafunctor (X, Φ) preserves the diagonal embeddings Δ if and only if X' preserves the diagonal embeddings Δ.*

Proof. Suppose first that X preserves the diagonal maps. The diagram

$$\begin{tikzcd} & X'\left(M^{\mathcal{U}}\right) \arrow[dd, equal] \\ X'(M) \arrow[ur, "X'(\Delta_M)"] \arrow[dr, "\Delta_{X'(M)}"'] & \\ & (X'(M))^{\mathcal{U}} \end{tikzcd}$$

commutes if and only if

$$X'(\Delta_M) = \Delta_{X'(M)}$$

if and only if (note that since M is being viewed as a trivial ultraproduct, $\Phi_M = \mathrm{id}_M$

and $\prod_{j \to \mathcal{V}} \sigma^M = \sigma^M$)

$$\begin{aligned} & X'(\Delta_M) \stackrel{?}{=} \Delta_{X'(M)} \\ \iff & \Phi'_{M^{\mathcal{U}}} \circ X(\Delta_M) \circ (\Phi'_M)^{-1} \stackrel{?}{=} \Delta_{X'(M)} \\ \iff & \prod_{i \to \mathcal{U}} \sigma_i^{M^{\mathcal{U}}} \circ \Phi_{M^{\mathcal{U}}} \circ X(\Delta_M) \circ \Phi_M^{-1} \circ \left(\sigma^M\right)^{-1} \stackrel{?}{=} \Delta_{X'(M)} \\ \iff & \prod_{i \to \mathcal{U}} \sigma_i^{M^{\mathcal{U}}} \circ \Delta_{X(M)} \circ \left(\sigma^M\right)^{-1} \stackrel{?}{=} \Delta_{X'(M)}. \end{aligned}$$

By chasing the diagram

$$\begin{tikzcd} X'(M) \arrow[d, "\Delta_{X'(M)}"'] & X(M) \arrow[l, "\sigma^M"'] \arrow[d, "\Delta_{X(M)}"] \\ X'(M)^{\mathcal{U}} & X(M)^{\mathcal{U}} \arrow[l, "\prod_{i \to \mathcal{U}} \sigma_i^M"] \end{tikzcd}$$

clockwise, we see that

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & \left(\sigma^M\right)^{-1} x \arrow[d, mapsto] \\ {[x]_{i \to \mathcal{U}}} & \left[\left(\sigma^M\right)^{-1} x\right]_{i \to \mathcal{U}} \arrow[l, mapsto] \end{tikzcd}$$

commutes, so the equation is true and X' preserves the diagonal maps.

Conversely, suppose X' preserves the diagonal maps. Then multiplying the left and right sides of the equation

$$\prod_{i \to \mathcal{U}} \sigma_i^{M^{\mathcal{U}}} \circ \Phi_{M^{\mathcal{U}}} \circ X(\Delta_M) \circ \Phi_M^{-1} \circ \left(\sigma^M\right)^{-1} = \Delta_{X'(M)}$$

by $\left(\prod_{i \to \mathcal{U}} \sigma_i^{M^{\mathcal{U}}}\right)^{-1}$ and σ^M, respectively, yields

$$\Phi_{(M)} \circ X(\Delta_M) = \left(\prod_{i \to \mathcal{U}} \sigma_i^{M^{\mathcal{U}}}\right)^{-1} \circ \Delta_{X'(M)} \circ \sigma^M \stackrel{?}{=} \Delta_{X(M)}.$$

Checking the final equality can be done by a diagram chase entirely analogous to the one from the first half of the proof. $\square$

Of course, the statement is true even when we replace Δ with general ultramorphisms.

Theorem 5.4.2. *Let δ and δ' be ultramorphisms of type $\langle \Gamma, k, \ell \rangle$ in the ultracategories $\mathbf{Mod}(T)$ and $\mathbf{Set}$, respectively. A pre-ultrafunctor $(X, \Phi) : \mathbf{Mod}(T) \to \mathbf{Set}$ carries δ into δ' if and only if the X' given by the construction 5.3.1 does also.*

Proof. Suppose first that X carries δ into δ'.

Let $\mathscr{M}$ be an ultradiagram in $\mathbf{Mod}(T)$. By the definition 3.3.16, X carries δ into δ' if and only if

$$\begin{tikzcd} X(\mathscr{M}(k)) \arrow[r, "X(\delta_{\mathscr{M}})"] \arrow[d, "\Phi_{\mathscr{M}(k)}"'] & X(\mathscr{M}(\ell)) \arrow[d, "\Phi_{\mathscr{M}(\ell)}"] \\ (X\mathscr{M})(k) \arrow[r, "\delta'_{X\mathscr{M}}"'] & (X\mathscr{M})(\ell) \end{tikzcd}$$

commutes. We need to check (whence strictness of X') that

$$X'(\delta_{\mathscr{M}}) \stackrel{?}{=} \delta'_{X'\mathscr{M}},$$

$$\begin{aligned} &\iff \Phi'_{\mathscr{M}(\ell)} \circ X(\delta_{\mathscr{M}}) \circ \left(\Phi'_{\mathscr{M}(k)}\right)^{-1} \stackrel{?}{=} \delta'_{X'\mathscr{M}} \\ \iff \prod_{i \to \mathcal{U}} \sigma_i^{\mathscr{M}(\ell)} \circ \Phi_{\mathscr{M}(\ell)} \circ X(\delta_{\mathscr{M}}) \circ \left(\Phi_{\mathscr{M}(k)}\right)^{-1} \circ &\left(\prod_{i \to \mathcal{U}} \sigma_i^{\mathscr{M}(k)}\right)^{-1} \stackrel{?}{=} \delta_{X'\mathscr{M}} \\ \iff \prod_{i \to \mathcal{U}} \sigma_i^{\mathscr{M}(\ell)} \circ \delta'_{X\mathscr{M}} \circ &\left(\prod_{i \to \mathcal{U}} \sigma_i^{\mathscr{M}(k)}\right)^{-1} \stackrel{?}{=} \delta'_{X'\mathscr{M}}, \end{aligned}$$

which is true if and only if the diagram

$$\begin{tikzcd} X\mathscr{M}(k) \arrow[r, "\delta'_{X\mathscr{M}}"] \arrow[d, "\prod_{i \to \mathcal{U}} \sigma_i^{\mathscr{M}(k)}"'] & X\mathscr{M}(\ell) \arrow[d, "\prod_{i \to \mathcal{U}} \sigma_i^{\mathscr{M}(\ell)}"] \\ X'\mathscr{M}(k) \arrow[r, "\delta'_{X'\mathscr{M}}"'] & X'\mathscr{M}(\ell) \end{tikzcd}$$

commutes. Since δ' is a natural transformation of the evaluation functors (k) and ℓ, to check that this square commutes, it suffices to check that the vertical maps arise from a morphism between the ultradiagrams $X\mathscr{M}$ and $X'\mathscr{M}$ in $\mathrm{Hom}(\Gamma, \mathbf{Set})$. The condition to check for this is

$$\Phi_\beta = \prod_{i \to \mathcal{U}_\beta} \Phi_{g_\beta(i)}$$

for all bound nodes β, but this is easily seen to be true after we remember that if k or ℓ are not bound, then their ultraproducts in the above square become trivial and $\prod_{i \to \mathcal{U}} \sigma_i^{\mathscr{M}(k)}$ is just a map $\sigma_{\mathscr{M}(k)}$.

Therefore, X' carries δ to δ'.

Conversely, suppose X' carries δ to δ'.

We need to check that

$$\delta'_{X\mathscr{M}} \stackrel{?}{=} \Phi_{\mathscr{M}(\ell)} \circ X(\delta_{\mathscr{M}}) \circ \left(\Phi_{\mathscr{M}(k)}\right)^{-1}.$$

Multiplying on the left by $\prod_{i \to \mathcal{U}} \sigma_i^{\mathscr{M}(\ell)}$ and on the right by $\left(\prod_{i \to \mathcal{U}} \sigma_i^{\mathscr{M}(k)}\right)^{-1}$, we get

$$\prod_{i \to \mathcal{U}} \sigma_i^{\mathscr{M}(\ell)} \circ \delta'_{X\mathscr{M}} \circ \left(\prod_{i \to \mathcal{U}} \sigma_i^{\mathscr{M}(k)}\right)^{-1} \stackrel{?}{=} X'(\delta_{\mathscr{M}}),$$

and by our assumption, the previous equation is true if and only if

$$\delta'_{X'\mathscr{M}} = X'(\delta_{\mathscr{M}}),$$

which is what we assumed. □

Chapter 6

Classifying toposes of first-order theories

The aim of this chapter and of chapter 7 is to prove that ultraproducts provide a natural characterization of the *coherent objects* of the classifying topos of a first-order theory. The classifying topos $\mathscr{E}(T)$ of T is a natural enlargement of $\mathbf{Def}(T)$ whose models in $\mathbf{Set}$ are the same as T's, and whose objects pick out a subcategory of evaluation functors $\mathbf{Mod}(T) \to \mathbf{Set}$ containing the image of $\mathrm{ev} : \mathbf{Def}(T) \to [\mathbf{Mod}(T), \mathbf{Set}]$. We will show in Theorem 7.2.1 that the property of ev_B being a pre-ultrafunctor with respect to a canonical transition map characterizes whether or not $B \in \mathscr{E}(T)$ is isomorphic to an object in $\mathbf{Def}(T)$.

6.1 Preliminaries on the classifying topos

For the construction and standard facts about the classifying topos of a first-order (or generally, a coherent) theory, see e.g. Part D of [7] or Volume III of [5]. For our convenience we will repeat the essentials for our results.

Throughout this chapter, "topos" will mean "Grothendieck topos", i.e. a category of sheaves on a small site. For detailed definitions of sites, sheaves, and toposes, we direct the reader to the relevant sections of the excellent references [10], [5], and [7].

For the reader's convenience, we will repeat Giraud's axiomatic characterization of Grothendieck toposes (see C.2.2.8, [7]):

Fact 6.1.1. *A (possibly large) category $\mathscr{E}$ is a Grothendieck topos if and only if the following conditions are satisfied:*

1. *Every class of morphisms $\mathscr{E}(X, Y)$ in $\mathscr{E}$ is a set ($\mathscr{E}$ is "locally small").*
2. *There exists a set $\mathcal{S}$ of objects in E such that for every pair of maps $f, g : X \to Y$ in $\mathscr{E}$ such that for all $S \in \mathcal{S}$, for all $e : S \to X$, $f \circ e = g \circ e$, then $e = g$ ($\mathscr{E}$ has a "small separating set of objects").*
3. *$\mathscr{E}$ has all small limits.*
4. *$\mathscr{E}$ has all small coproducts, which are disjoint and stable under pullback.*
5. *All equivalence relations in $\mathscr{E}$ have quotients which are stable under pullback.*

Note the similary to the definition of a pretopos (Definition 2.6.16). Indeed, it was shown in [13] that one could generalize the closure under "finitary" operations defining a pretopos to a notion of a κ-pretopos for a regular cardinal κ, and that Grothendieck toposes are precisely ∞-pretoposes with a small separating set of objects.

Definition 6.1.2. The **classifying topos** of a first-order theory T is a topos $\mathscr{E}(T)$ equipped with a fully faithful functor $\mathbf{y} : \mathbf{Def}(T) \to \mathscr{E}(T)$ which is also a model in the sense of 2.5.1 (the definition given there only involves the preservation of certain categorical properties, so makes sense for functors into any topos instead of **Set**). $\mathscr{E}(T)$ additionally satisfies the following universal property: for any other topos $\mathscr{S}$ and any model $M : \mathbf{Def}(T) \to \mathscr{S}$ of $\mathbf{Def}(T)$ in $\mathscr{S}$, there exists a unique $\widetilde{M} : \mathscr{E}(T) \to \mathscr{S}$ such that the diagram

$$\begin{tikzcd} \mathscr{E}(T) \arrow[dr, "\widetilde{M}"] & \\ \mathbf{Def}(T) \arrow[u, "\mathbf{y}"] \arrow[r, "M"'] & \mathscr{S} \end{tikzcd}$$

commutes.

This characterizes $\mathscr{E}(T)$ up to equivalence. We call $\widetilde{M}$ the **inverse image functor** associated to the model M. We also call objects of $\mathscr{E}(T)$ which are, up to isomorphism, in the image of $\mathbf{y}$ **representable** (echoing the standard construction of $\mathscr{E}(T)$ as a certain category of sheaves on $\mathbf{Def}(T)$.)

As the definition indicates, the extension $\widetilde{M}$ of M from $\mathbf{Def}(T)$ to $\mathscr{E}(T)$ should be determined by what M does on the objects of $\mathbf{Def}(T)$. The following discussion is meant to make this intuition explicit, and to give a formula for computing what $\widetilde{M}$ is outside of the image of $\mathbf{y}$ inside $\mathscr{E}(T)$.

6.1.1 Computing the associated inverse image functor $\widetilde{M}$

Definition 6.1.3. (3.7.1 of [5]) Let $F : A \to B$ and $G : A \to C$ be functors. The **left Kan extension** of G along F, if it exists, is a pair (K, α) where $K : B \to C$ is a functor and $\alpha : G \to K \circ F$ is a natural transformation satisfying the following universal property if (H, β) is another pair with $H : B \to C$ a functor and $\beta : G \to H \circ F$ a natural transformation, then there exists a unique natural transformation $\gamma : K \to H$ satisfying the equality $(\gamma F) \circ \alpha = \beta$, as in the following diagram:

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "G"'] & B \arrow[dl, shift right, "K"'] \arrow[dl, shift left, "H"] \\ C & \end{tikzcd} \quad , \quad \gamma : K \xrightarrow{!} H.$$

We write $\mathrm{Lan}_F\, G$ for the left Kan extension of G along F. Right Kan extensions are defined dually, and are written $\mathrm{Ran}_F\, G$.

Before proceeding, we give two definitions around the category of points of a (contravariant) functor.

Definition 6.1.4. Consider the diagram of functors
$$\begin{tikzcd} C \arrow[dr, "F"'] & & D. \arrow[dl, "G"] \\ & E & \end{tikzcd}$$
The **comma category** $(F \downarrow G)$ is given by:

Objects: (c, d, α) where $c \in C, d \in D, \alpha : F(c) \to G(d) \in E$.

Morphisms: $\mathrm{Hom}_{(F\downarrow G)}\left((c_1, d_1, \alpha_1), (c_2, d_2, \alpha_2)\right)$ is defined to be the set

$$\left\{ (\beta_1, \beta_2) \,\middle|\, \beta_1 : c_1 \to c_2, \beta_2 : d_1 \to d_2, \text{ and } \begin{tikzcd} F(c_1) \arrow[r, "F(\beta_1)"] \arrow[d, "\alpha_1"'] & F(c_2) \arrow[d, "\alpha_2"] \\ G(d_1) \arrow[r, "G(\beta_2)"] & G(d_2) \end{tikzcd} \text{ commutes.} \right\}$$
.

Definition 6.1.5. If $F \colon C \to \mathbf{Set}$ is a $\mathbf{Set}$-valued functor on a locally small category C, the **category of (global) points of** F, written $\int^{c\in C} F(c)$, is the comma category $(1 \downarrow F)$.

Explicitly, it is given by:

Objects: $\left\{(c, x) \,\middle|\, c \in C, x \in F(C)\right\}$.

Morphisms: $\mathrm{Hom}_{\int^{c\in C} F(c)}\left((c_1, x_1), (c_2, x_2)\right)$ is defined to be the set

$$\left\{ f \,\middle|\, f : c_1 \to c_2 \text{ and } F(f)(x_1) = x_2. \right\}$$

If $F : C^{\mathrm{op}} \to D$ is a contravariant functor, we write $\int_{c\in C} F(c)$ for the opposite of $\int^{c\in C} F(c)$.

The category of points of a functor $F : C \to D$ is equipped with a projection (forgetful) functor π back to C.

Lemma 6.1.6. *(3.7.2 of [5]) Consider two functors $F : A \to B$ and $G : A \to C$ with A small and C cocomplete. Then the left Kan extension of G along F exists, and is* *given pointwise by a colimit*

$$(b \to b') \mapsto \varinjlim\left(\int^{a \in A} B(a,b) \xrightarrow{\pi} A \xrightarrow{G} C\right) \to \varinjlim\left(\int^{a \in A} B(a,b') \xrightarrow{\pi} A \xrightarrow{G} C\right)$$

Lemma 6.1.7. *(3.7.3 of [5]) Let $F : A \to B$ be a full and faithful functor with A a small category. Let C be a cocomplete category. Then for any functor $A \to C$, the canonical natural transformation $G \xrightarrow{\alpha} (\operatorname{Lan}_F G) \circ F$ is an isomorphism (so that the inner triangle from 6.1.3 "commutes").*

Corollary 6.1.8. *Every model $M : \mathbf{Def}(T) \to \mathbf{Set}$ extends uniquely along $\mathbf{y}$ $\mathbf{Def}(T) \xhookrightarrow{\mathbf{y}} \mathscr{E}(T)$ to an inverse image functor $\widetilde{M}$, as in*

$$\begin{tikzcd} \mathscr{E}(T) \arrow[dr, "\widetilde{M}"] & \\ \mathbf{Def}(T) \arrow[u, hook, "\mathbf{y}"] \arrow[r, "M"'] & \mathbf{Set} \end{tikzcd} .$$

The extension to $\mathscr{E}(T)$ is given by a pointwise Kan extension, so that for any $B \in \mathscr{E}(T)$, $\widetilde{M}(B)$ can be computed as the colimit

$$\varinjlim\left(\int^{A \in \mathbf{Def}(T)} \mathscr{E}(T)(A,B) \xrightarrow{\pi} \mathbf{Def}(T) \xrightarrow{M} \mathbf{Set}\right).$$

6.2 Coherence, compactness and definability in $\mathscr{E}(T)$

In this section, we review the necessary parts of the theory of classifying toposes of first-order theories. We refer the reader to section D3 of [7] for details.

Definition 6.2.1. An object A of a topos $\mathscr{E}$ is **compact** if every covering family of maps $\{f_i \mid i \in I\}$ of maps into A contains a finite subcover.

Definition 6.2.2. An object A of a topos $\mathscr{E}$ is **stable** if for every morphism $f : B \to A$ where B is compact, the domain K of the kernel relation $K \rightrightarrows B \xrightarrow{f} A$ is also compact.

Definition 6.2.3. An object A of a topos $\mathscr{E}$ is **coherent** if it is both compact and stable.

Remark 6.2.4. In a coherent topos, the pretopos of coherent objects is not necessarily closed under arbitrary finite colimits. This is because coequalizers are quotients by (at least) transitive closures of certain relations, so if one has a relation $R \rightrightarrows X$ whose transitive closure is properly ind-definable, the coequalizer $\mathbf{y}(R) \rightrightarrows \mathbf{y}(X) \twoheadrightarrow Y$ will not be definable.

Lemma 6.2.5. *(D3.3.7, [7]) An object B of the classifying topos $\mathscr{E}(T)$ of a first-order theory T is representable (i.e. isomorphic to an object from $\mathbf{Def}(T) \hookrightarrow \mathscr{E}(T)$) if and only if it is coherent.*

Remark 6.2.6. If one constructs the classifying topos $\mathscr{E}(T)$ as a category of sheaves on $\mathbf{Def}(T)$ (where T might not necessarily eliminate imaginaries), then taking the coherent objects of $\mathscr{E}(T)$ yields an alternate construction of the pretopos completion of $\mathbf{Def}(T)$. Thus, if T eliminates imaginaries (as we have assumed for most of this document), the pretopos completion of $\mathbf{Def}(T)$ is isomorphic to $\mathbf{Def}(T)$, hence the previous lemma.

Notation 6.2.7. From now on, when working in the classifying topos $\mathscr{E}(T)$ of a first-order theory, we will use "definable" and "coherent" interchangably.

Definition 6.2.8. Let $\mathbf{C}$ be a category, and let B be an object of $\mathbf{C}$. Let c_B be the constant functor $\mathbf{C} \to B$ which sends every morphism in $\mathbf{C}$ to id_B. The **slice category** $\mathbf{C}/B$ is defined to be the comma category (Definition 6.1.4) $(\mathrm{id}_{\mathbf{C}}, c_B)$.

The "fundamental theorem of topos theory" (see A2.3, [7]) says that for a topos $\mathscr{E}$, *any* slice category $\mathscr{E}/B$ of $\mathscr{E}$ is also a topos.

Lemma 6.2.9. *(D3.3.16, [7]) Let $\mathscr{E}(T)$ be the classifying topos for a first-order theory. Then an object $B \in \mathscr{E}(T)$ is coherent if and only if the slice category $\mathscr{E}(T)/B$ is a coherent topos, which is presented by the coherent site $\mathbf{Def}(T)/B$ of coherent objects $A \in \mathscr{E}(T)$ over B.*

We also record the following observation, which seems to be folklore:

Lemma 6.2.10. *Let $B \in \mathscr{E}(T)$ be coherent. Then the slice topos $\mathscr{E}(T)/B$ is presented by the theory of T extended by a generic constant of b, written $T[b : B]$.*

Proof. One easily verifies that the adjoint $(-) \times B$ to the forgetful functor $\mathscr{E}(T)/B \to \mathscr{E}(T)$ restricts to an interpretation of the underlying pretoposes which factors through $\mathbf{Def}(T[b : B])$, and that the induced map between the categories of models $\mathbf{Mod}(T[b : B]) \leftarrow \mathbf{Mod}(\mathscr{E}(T)/B)$ is an isomorphism. By conceptual completeness 3.2.11, the map $\mathbf{Def}(T[b : B]) \to \mathbf{Def}(T)/B$ was a bi-interpretation of pretoposes. $\square$

Chapter 7

Ultraproducts and coherence in the classifying topos

7.1 Compact non-coherent objects in $\mathscr{E}(T)$

In the previous chapter, we introduced the notions of compact, stable, and coherent objects in $\mathscr{E}(T)$, and we claimed that the coherent objects were precisely the definable ones. In this section, we analyze the compact non-coherent objects. As we saw in 6.2.4, the prototypical example in a coherent topos of a compact non-coherent object is the coequalizer of a definable relation $R \rightrightarrows X$ on a definable set X with a properly ind-definable transitive closure. Our aim in this section is to prove the lemma 7.1.4, which says that this obstruction to coherence actually characterizes the compact non-coherent objects in a coherent topos.

An important basic category-theoretic fact is the canonical coproduct-coequalizer decomposition of colimits (whose proof can be found, for example, in [9]):

Fact 7.1.1. *Let $\mathcal{D}$ be a subcategory of* $\mathbf{C}$ *a category with all colimits.*

Then the colimit $\varinjlim(\mathcal{D})$ *of* $\mathcal{D}$ *is isomorphic to the coequalizer of the following diagram:*

$$\left(\coprod_{f \in \mathcal{D}_1} s(f) \right) \overset{F}{\underset{G}{\rightrightarrows}} \left(\coprod_{d \in \mathcal{D}_0} d \right)$$

where on each component $s(f) \in \mathcal{D}_0$ *of the left hand side,* F *sends* $s(f)$ *to itself* $d = s(f)$ *by the identity map of* $d = s(f)$*, and on each* $s(f) \in \mathcal{D}_0$ *of the left hand side,* G *sends* $s(f)$ *to* $t(f)$ *by the map* f*.*

We apply this fact to show the following:

Lemma 7.1.2. *An object* B *of a coherent topos* $\mathscr{E}(T)$ *is compact if and only if every covering of* B *whose domains are representables admits a finite subcover.*

Proof. The implication "$\Rightarrow$" is immediate.

Conversely, suppose that $\{B_i \to B\}$ is a covering of B. By the Kan extension colimit formula and the coproduct-coequalizer decomposition of colimits, each B_i is covered by (possibly infinitely many) representables. The collection of all these representables across all B_i form a covering of representables of B. By assumption, this covering admits a finite subcovering. Therefore, only finitely many of these B_i were needed since all these representable coverings factored through some B_i. $\square$

We recount the following fact from [6], closely related to the lemma 7.1.4:

Fact 7.1.3. *(Lemma 7.36 of [6]). Let* $\mathscr{E}$ *be a topos generated by compact objects. Let* X *be a coherent object of* E*, and let* $R \rightrightarrows X$ *be an equivalence relation with coequalizer* $R \rightrightarrows X \twoheadrightarrow X$*.*

Then Y *is coherent if and only if* R *is compact.*

Our next lemma 7.1.4 is a sharpening of the fact 7.1.3: not only will we show that a compact non-coherent object is the quotient of a coherent object by a non-compact congruence, but we will explicitly describe the non-compact congruence as an infinite join of coherent objects.

Lemma 7.1.4. *Let $B \in \mathscr{E}(T)$ be a compact non-coherent object. Then B is the quotient of a coherent object A by a non-compact equivalence relation E which is a join of infinitely many coherent equivalence relations on A.*

Proof. Write B as a colimit of a diagram $\mathcal{D}$ whose objects are representables A_i. By the coproduct-coequalizer decomposition, B is a quotient of the coproduct $\bigsqcup_{A \in \mathcal{D}_i} A$ and therefore the maps $A_i \hookrightarrow \bigsqcup_{A \in \mathcal{D}_i} A \overset{p_B}{\to} B$ are a covering family for B. Since B is compact, finitely many A_i, say $A_1, \ldots, A_n$ suffice to cover B.

What we have said so far amounts to saying that B is a quotient of the coherent object $\bigsqcup_{i \leqslant n} A_i$, since the obvious map

$$\left(\bigsqcup_{i \leqslant n} A_i\right) \overset{i}{\hookrightarrow} \left(\bigsqcup_{A \in \mathcal{D}_0} A\right) \overset{p_B}{\to} B$$

covers B.

It now remains to calculate the kernel relation K' of $p_B \circ i$ and show that it is an infinite union of coherent relations on $\bigsqcup_{i \leqslant n} A_i$.

We break the remainder of the proof into the following steps:

1. The kernel relation K' of $p_B \circ i$ is the pullback of the kernel relation K of p_B along the inclusion

$$i \times i : \left(\bigsqcup_{i \leqslant n} A_i\right) \times \left(\bigsqcup_{i \leqslant n} A_i\right) \hookrightarrow \left(\bigsqcup_{A \in \mathcal{D}_0} A\right) \times \left(\bigsqcup_{A \in \mathcal{D}_0} A\right)$$

and therefore in every model consists of those pairs $(a_1, a_2) \in K$ such that both a_1 and a_2 are in $\bigsqcup_{i \leqslant n} A_i$.

2. Fix an arbitrary model. There is no harm in working with points and sets in a generic model since by Deligne's completeness theorem we can then lift our calculations to the classifying topos.

Now, K is by definition the smallest equivalence relation containing "$\exists b : F(b) = a_1$ and $G(b) = a_2 \implies a_1 \sim_K a_2$." By how F and G are constructed, this means that $a \sim_K a'$ if and only if there are finitely many other points $a_1, \ldots, a_n$ and maps linking a to a_1, each a_i to a_{i+1}, and a_n to a', where the maps may point in either direction.

It follows that K' is finer than just the kernel relation of the coequalizer of the pullback of $F, G : \bigsqcup_{A \in \mathcal{D}_0} A \to \bigsqcup_{A \in \mathcal{D}_0} A$ along the inclusion i, and is given by the following union:

$$K' = \bigvee_{n \in \omega} R_n$$

where R_0 is the diagonal copy of $\bigsqcup_{i \leqslant n} A_i$, R_1 consists of those pairs (a_1, a_2) such that there is some a_0' in $\bigsqcup_{A \in \mathcal{D}_0} A$ such that there is a map f in $\mathcal{D}_1$ that moves a_1 to a_0' or vice-versa, and there is a map g in $\mathcal{D}_1$ that moves a_0' to a_2 or vice-versa, etc.

3. R_1 is the infinite union $\bigvee_{A \in \mathcal{D}_0} S_A$, where each S_{A_k} corresponds to the A containing a particular witness $a_k = a_0'$ as above.

4. Each S_{A_k} looks like this:

$$\bigvee_{(f,f',g,g')} \left\{ (a_i, a_j) \in A_i \times A_j \,\middle|\, \exists a_k \in A_k \big((a_i, a_k) \in \Gamma(f) \vee \Gamma(f') \text{ and } (a_j, a_k) \in \Gamma(g) \vee \Gamma(g') \big) \right\},$$

where the 4-tuple of maps (f, f', g, g') ranges over definable maps

$$\mathbf{Def}(T)(A_i, A_k) \times \mathbf{Def}(T)(A_k, A_i) \times \mathbf{Def}(T)(A_j, A_k) \times \mathbf{Def}(T)(A_k, A_j)$$

and therefore each S_{A_k} is $\bigvee$-coherent.

Therefore, R_1 is $\bigvee$-coherent.

5. Let us inductively assume that R_k is $\bigvee$-coherent as the union $\bigvee_{i \in I} T_i$. Then R_{k+1} is the following subset of $R_k \times R_1$:

$$R_{k+1} = \left\{ (a, b) \,\middle|\, \bigvee_{(T_i, S_A) \in I \times \mathcal{D}_0} \exists c \ \text{ s.t. } (a, c) \in T_i \wedge (a, b) \in S_A \right\}$$

and is therefore also $\bigvee$-coherent.

We conclude that K' is $\bigvee$-coherent. □

7.2 The coherence criterion

Theorem 7.2.1. *Let $\mathscr{E}(T)$ be the classifying topos of a first-order theory. Let B be an object of $\mathscr{E}(T)$. The following are equivalent:*

1. *B is coherent.*
2. $\mathrm{ev}_B : \mathbf{Mod}(T) \to \mathbf{Set}$ *is the underlying functor of a pre-ultrafunctor* (ev_B, Φ) *such that, if B is canonically the colimit of representables A_i, then each canonical map $A_i \to B$ induces an ultratransformation of the pre-ultrafunctors* $(\mathrm{ev}_{A_i}, \mathrm{id}) \to (\mathrm{ev}_B, \Phi)$.

Proof. ($1 \implies 2$) If B is coherent, then it is representable and $(\mathrm{ev}_B, \mathrm{id})$ is a pre-ultrafunctor,, and since $\mathbf{y} : \mathbf{Def}(T) \to \mathscr{E}(T)$ is full and faithful, every map $A_i \to B$ corresponds to a definable function, which induces an ultratransformation $\mathrm{ev}(A_i) \to \mathrm{ev}(B)$.

($2 \implies 1$) First, we note that under the assumptions, ev_B's transition isomorphism is uniquely determined by the transition isomorphisms of the representables appearing in the Kan extension colimit formula for B: all diagrams of the form

$$\begin{tikzcd}
 & \mathrm{ev}_B\left(\prod_{i \to \mathcal{U}} M_i\right) \arrow[rr, "\Phi^B_{(M_i)}"] & & \prod_{i \to \mathcal{U}} \mathrm{ev}_B(M_i) \\
\mathrm{ev}_A\left(\prod_{i \to \mathcal{U}} M_i\right) \arrow[ur] \arrow[dr] \arrow[rr, "\Phi^A_{(M_i)}"'] & & \prod_{i \to \mathcal{U}} \mathrm{ev}_A(M_i) \arrow[ur] \arrow[dr] & \\
 & \mathrm{ev}_{A'} \arrow[uu] \arrow[rr, "\Phi^{A'}_{(M_i)}"'] & & \prod_{i \to \mathcal{U}} \mathrm{ev}_{A'}(M_i) \arrow[uu]
\end{tikzcd}$$

commute, and since the Kan extension colimit formula is computed pointwise, the transition isomorphism $\Phi^B_{(M_i)}$ is a unique comparison map from the colimit $\mathrm{ev}_B\left(\prod_{i \to \mathcal{U}} M_i\right)$ of the $\mathrm{ev}_A(\prod_{i\to\mathcal{U}} M_i)$'s into $\prod_{i \to \mathcal{U}} \mathrm{ev}_B(M_i)$.

Now, knowing this, suppose B is not coherent. Then either B cannot be covered by finitely many definables, or it can. If it can be covered by the finitely many definables $A_1, \ldots, A_n$, then the associated map $A_1 \sqcup \cdots \sqcup A_n \twoheadrightarrow B$ does not have a definable kernel relation, and in fact by 7.1.4, the kernel relation is properly ind-definable.

In either case, we know what the transition isomorphism $\Phi^B_{(M_i)}$ looks like. In the first case, if B cannot be covered by finitely many definables, we still know from the Kan extension colimit formula that it can be covered by infinitely many $(A_i)_{i \in I}$. Fix a model M and take a sequence $(a_i)_{i \in I}$ such that for every A_j, cofinitely many a_i are not in (the image of) A_j (in B). Then for a non-principal ultrafilter $\mathcal{U}$ on I, $[a_i]_{i \in \mathcal{U}}$ is not in any of the (images of the) $(M^{\mathcal{U}})(A_j)$'s. Therefore, it is not in the image of the transition isomorphism $\Phi^B_{(M)}$, a contradiction.

In the second case, if B looks like a definable set A quotiented by a properly ind-definable equivalence relation $R = \bigcup_{i \in I} R_i$, then once again we know that the transition isomorphism

$$\left(\prod_{i \to \mathcal{U}} M_i\right)(A/R) \to \prod_{i \to \mathcal{U}} (M_i(A/R))$$

is the "obvious" one. Here's what the "obvious" map is: since A is definable, we are really comparing two equivalence relations on the same set. On the left hand side, we have that $[a_i]_{i \to \mathcal{U}} \sim [b_i]_{i \to \mathcal{U}}$ if and only if there exists some R_j such that $(\prod_{i \to \mathcal{U}} M_i)(R_j)$ contains $([a_i], [b_i])_{i \to \mathcal{U}}$. On the right hand side, we have that $[a_i]_{i \to \mathcal{U}} \sim [b_i]_{i \to \mathcal{U}}$ if and only if $a_i \sim_R b_i$ $\mathcal{U}$-often. Since R is properly ind-definable, the equivalence relation on the left is properly contained in the equivalence relation on the right. This containment induces a map between the quotients, and since the containment is proper, this map is not injective, and cannot be a bijection.

□

Now we use this result to prove a stronger statement than 4.3.2. The difference is that in the original statement of 4.3.2, we only concluded that X was definable, without saying anything about the transition isomorphism Φ which allowed us to view (X, Φ) as a Δ-functor. In fact, we can show that (X, Φ) is isomorphic to $\mathrm{ev}_{\varphi(x)}$, and must therefore be an ultrafunctor.

Theorem 7.2.2. *Let T be $\aleph_0$-categorical. Let (X, Φ) be a pre-ultrafunctor. Then the underlying functor X is definable if and only if for some $\varphi(x) \in T$, (X, Φ) is isomorphic as a pre-ultrafunctor to $\mathrm{ev}_{\varphi(x)}$ (equivalently, by strong conceptual completeness 3.5.1, (X, Φ) is an ultrafunctor).*

Proof. By applying the lemma 4.3.1 that Δ-functors preserve filtered colimits and arguing as in the first part of the proof of 4.3.2, we conclude that X is isomorphic to a possibly infinite disjoint union of representables $\bigsqcup_{i \in I} A_i$. In this way, X is canonically the colimit of the representables A_i. It remains to verify the rest of item 2, i.e. the canonical inclusions $A_k \hookrightarrow \bigsqcup_{i \in I} A_i \simeq X$ induce ultratransformations.

Before proceeding, we reduce the problem of verifying this for all ultraproducts to just verifying this for all ultrapowers. This is because, in general, every ultraproduct is a filtered colimit of ultraproducts of countable models: for every $[x_i]_{i \to \mathcal{U}}$ in some ultraproduct $\prod_{i \to \mathcal{U}} N_i$, take a countable elementary model $M_i \overset{f_i}{\hookrightarrow} N_i$ which contains x_i; then there is an embedding $\prod_{i \to \mathcal{U}} f_i : \prod_{i \to \mathcal{U}} M_i \hookrightarrow \prod_{i \to \mathcal{U}} N_i$, and the collection of all such embeddings covers $\prod_{i \to \mathcal{U}} N_i$. Since T is $\aleph_0$-categorical, an ultraproduct of countable models is just an ultrapower of the unique countable model.

So, it remains to check that the diagram

$$\begin{tikzcd} X\left(M^{\mathcal{U}}\right) \arrow[rr, "\Phi_{(M)}"] & & X(M)^{\mathcal{U}} \\ & A\left(M^{\mathcal{U}}\right) \arrow[ul, "\iota_{M^{\mathcal{U}}}"] \arrow[ur, "\prod_{i \to \mathcal{U}} \iota_i"'] & \end{tikzcd}$$

commutes. Each component ι_N of the ultratransformation is determined by filtered colimits of the countable model M, with ι_M determined by sending the support $a_x \in A(M)$ to x. Since $\Delta_M : M \to M^{\mathcal{U}}$ is part of the filtered diagram of countable submodels of $M^{\mathcal{U}}$, $\iota_{M^{\mathcal{U}}}$ of $\Delta_M(a_x) = X(\Delta_M)(x)$, and since (X, Φ) was a Δ-functor, $\Phi_{(M)} \circ X(\Delta_M)(x) = \Delta_{X(M)}(x)$.

On the other hand,

$$\prod_{i \to \mathcal{U}} \iota_i \left(\Delta(a_x)\right) = [\iota_M(a_x)]_{i \to \mathcal{U}} = \Delta_{X(M)}(x).$$

So the diagram commutes, and now we are done by the direction $2 \implies 1$ of the theorem. $\square$

Chapter 8

Exotic functors $\mathbf{Mod}(T) \to \mathbf{Set}$

In this chapter, we will construct for, certain theories T, "exotic" functors $\mathbf{Mod}(T) \to \mathbf{Set}$ which will exhibit the failure of 7.2.2 when the assumption of $\aleph_0$-categoricity is removed.

8.1 Counterexamples to Theorem 7.2.2 in the non-$\aleph_0$-categorical case

The basis for our counterexamples is the theory of an infinite set, expanded by countably many distinct constants. We will construct an example of a pre-ultrafunctor which is not a Δ-functor, and an example of a Δ-functor which is not an ultrafunctor (specifically, we will find an example which fails to preserve the generalized diagonal embeddings 3.3.13).

For the rest of this section, T will mean the theory of an infinite set with countable many distinct constants $\{c_i\}_{i \in \omega}$. In a single variable, T has a unique non-isolated type $p(x)$, whose realizations are those elements which are not any constants.

Definition 8.1.1. The underlying functor X for the pre-ultrafunctors we will construct will be given on the objects of $\mathbf{Mod}(T)$ by:

$$X(M) \overset{\text{df}}{=} p(M) \cup \left\{ c_k^M \,\middle|\, k \text{ is even} \right\}.$$

On elementary embeddings $f : M \to N$, we set $X(f)$ to just be the restriction of f to $X(M)$.

There is an obvious map which compares $\prod_{i \to \mathcal{U}} X(M_i)$ with $X\left(\prod_{i \to \mathcal{U}} M_i\right)$, namely the inclusion of the former in the latter. However, this cannot be an isomorphism, since any unbounded increasing sequence of odd constants will realize p in an ultrapower. To complete the construction of the counterexamples, it remains to construct transition isomorphisms for X.

For our convenience, we record an analysis of the automorphisms of the functor X which will be useful in the construction of the exotic Δ-functor 8.1.1.

Lemma 8.1.2. *Any automorphism $\eta : X \to X$ of X satisfies the following property: for every $M \models T$, $\eta_M : X(M) \to X(M)$ permutes the constants and fixes the nonconstants.*

Proof. Fix an arbitrary model M, let $\Delta_M : M \to M^U$ be the diagonal embedding into some ultrapower $M^{\mathcal{U}}$, and consider the naturality diagram which must be satisfied by the components $\{\eta_M\}_{M \in \mathbf{Mod}(T)}$ of η:

$$\begin{tikzcd}
M \arrow[d, "\Delta_M"'] & X(M) \arrow[r, "\eta_M"] \arrow[d, "X(\Delta_M)"'] & X(M) \arrow[d, "X(\Delta_M)"] \\
M^{\mathcal{U}} & X\left(M^{\mathcal{U}}\right) \arrow[r, "\eta_{M^{\mathcal{U}}}"'] & X\left(M^{\mathcal{U}}\right)
\end{tikzcd}$$

Suppose η_M sends a constant c to a nonconstant $\eta_M(c)$. Then the commutativity of the naturality diagram tells us $\eta_{M^{\mathcal{U}}}$ sends $X(\Delta_M)(c) = \Delta_M(c)$ to $X(\Delta_M)(\eta_M(c)) =$

$\Delta_M(\eta_M(c))$. However, any injection $M \to M^U$ which identifies constants with constants and sends nonconstants to nonconstants is an elementary embedding, and we can certainly find an embedding $f : M \to M^U$ which does not send the nonconstant $\eta_M(c)$ to $\Delta_M(\eta_M(c))$. Then, since elementary embeddings fix constants, the naturality diagram

$$\begin{tikzcd} M \arrow[d, "f"'] & X(M) \arrow[r, "\eta_M"] \arrow[d, "X(f)"'] & X(M) \arrow[d, "X(f)"] \\ M^{\mathcal{U}} & X\left(M^{\mathcal{U}}\right) \arrow[r, "\eta_{M^{\mathcal{U}}}"'] & X\left(M^{\mathcal{U}}\right) \end{tikzcd}$$

would not commute. So, η_M must send constants to constants. Since η is an isomorphism and hence invertible, η_M cannot send nonconstants to constants either.

Now suppose that η_M does not fix the nonconstants, so that for some nonconstant d, $d \neq \eta_M(d)$, with $\eta_M(d)$ a nonconstant. Consider again the naturality diagram for $\Delta_M : M \to M^{\mathcal{U}}$:

$$\begin{tikzcd} M \arrow[d, "\Delta_M"'] & X(M) \arrow[r, "\eta_M"] \arrow[d, "X(\Delta_M)"'] & X(M) \arrow[d, "X(\Delta_M)"] \\ M^{\mathcal{U}} & X\left(M^{\mathcal{U}}\right) \arrow[r, "\eta_{M^{\mathcal{U}}}"'] & X\left(M^{\mathcal{U}}\right) \end{tikzcd}$$

This tells us that $\eta_{M^{\mathcal{U}}}(\Delta_M(d)) = \Delta_M(\eta_M(d))$.

Let d' stand for $\Delta_M(\eta_M(d))$, and let e be another nonconstant in $M^{\mathcal{U}}$, distinct from $\Delta_M(d)$ and d'. Since d' and e are nonconstants, we can find an automorphism $\sigma : M^{\mathcal{U}} \to M^{\mathcal{U}}$ which fixes $\Delta_M(d)$ but which moves d' to e. Then the naturality diagram for σ

$$\begin{tikzcd} M^{\mathcal{U}} \arrow[d, "\sigma"'] & X\left(M^{\mathcal{U}}\right) \arrow[r, "\eta_{M^{\mathcal{U}}}"] \arrow[d, "X(\sigma)"'] & X\left(M^{\mathcal{U}}\right) \arrow[d, "X(\sigma)"] \\ M^{\mathcal{U}} & X\left(M^{\mathcal{U}}\right) \arrow[r, "\eta_{M^{\mathcal{U}}}"'] & X\left(M^{\mathcal{U}}\right) \end{tikzcd}$$

tells us that

$$\begin{aligned}\sigma \circ \eta_{M^{\mathcal{U}}}(\Delta_M(d)) &= \eta_{M^{\mathcal{U}}} \circ \sigma(\Delta_M(d)) \\ &= \sigma(d') = \eta_M^{\mathcal{U}}(\Delta_M(d)) \\ &= e = d',\end{aligned}$$

a contradiction. Therefore, η_M fixes the nonconstants. □

Finally, we remark that *any* permutation of the constants can be realized in an automorphism $\eta : X \to X$, and in fact $\mathrm{Aut}(X) \simeq \mathrm{Sym}(\omega)$.

8.1.1 The exotic Δ-functor

Now we will construct a transition isomorphism Φ for X such that (X, Φ) is a Δ-functor which is not an ultrafunctor (and, in fact, which fails to preserve the generalized diagonal embeddings 3.3.13).

Fix I and a non-principal ultrafilter $\mathcal{U}$. Let $(M_i)_{i \in I}$ be an I-indexed sequence of models. Consider $X\left(\prod_{i \to \mathcal{U}} M_i\right)$, in which we can canonically identify $\prod_{i \to \mathcal{U}} X(M_i)$ as a subset.

Definition 8.1.3. Let $A_{(M_i)}$ be the complement of $\prod_{i \to \mathcal{U}} X(M_i)$ inside $X\left(\prod_{i \to \mathcal{U}} M_i\right)$. $A_{(M_i)}$ consists of those elements $[x_i]_{i \to \mathcal{U}}$ of $\prod_{i \to \mathcal{U}} M_i$ which:

1. realize the non-isolated type $p(x)$, i.e. are not constants, and
2. such that any representative sequence $(x_i)_{i \to \mathcal{U}}$ is $\mathcal{U}$-often an odd constant (equivalently, can be represented by a sequence made up entirely of odd constants).

Let $B_{(M_i)}$ be the subset of $\prod_{i \to \mathcal{U}} X(M_i)$ which consists of those elements $[x_i]_{i \to \mathcal{U}}$ of $\prod_{i \to \mathcal{U}}$ which:

1. realize the non-isolated type $p(x)$, i.e. are not constants, and

2. such that any representative sequence $(x_i)_{i\to\mathcal{U}}$ is $\mathcal{U}$-often an even constant (equivalently, can be represented by a sequence made up entirely of even constants).

Finally, let $C_{(M_i)}$ be the complement of $B_{(M_i)}$ inside $\prod_{i\to\mathcal{U}} X(M_i)$.

Note that $C_{(M_i)}$ consists precisely of those elements of $X\left(\prod_{i\to\mathcal{U}} M_i\right)$ which are either constants or which are nonconstants $[x_i]_{i\to\mathcal{U}}$ for which any representative sequence $(x_i)_{i\to\mathcal{U}}$ is $\mathcal{U}$-often a nonconstant.

Since elementary embeddings preserve the property of a tuple being constant or nonconstant, for any sequence of elementary embeddings $(f_i : M_i \to N_i)_{i\to\mathcal{U}}$, we have that $[f_i]_{i\to\mathcal{U}}$ restricts to a map $C_{(M_i)} \to C_{(N_i)}$, and furthermore because elementary embeddings fix constants, $[f_i]_{i\to\mathcal{U}}$ restricts to bijections $A_{(M_i)} \to A_{(N_i)}$ and $B_{(M_i)} \to B_{(N_i)}$.

Now, we have disjoint unions

$$X\left(\prod_{i\to\mathcal{U}} M_i\right) = A_{(M_i)} \sqcup B_{(M_i)} \sqcup C_{(M_i)} \quad \text{and} \quad \prod_{i\to\mathcal{U}} X(M_i) = B_{(M_i)} \sqcup C_{(M_i)},$$

and our task is to find a transition isomorphism $\Phi_{(M_i)} : A_{(M_i)} \sqcup B_{(M_i)} \sqcup C_{(M_i)} \xrightarrow{\sim} B_{(M_i)} \sqcup C_{(M_i)}$.

We define $\Phi_{(M_i)}$ to be the identity on $C_{(M_i)}$. It remains to specify a bijection $\sigma : A_{(M_i)} \sqcup B_{(M_i)} \simeq B_{(M_i)}$. Since any such σ only involves identifying certain ultraproducts of constants with other ultraproducts of constants, then after fixing a σ we can use σ to define $\Phi_{(N_i)}$ for arbitrary I-indexed sequences of models (N_i). With this setup, we will show that any choice of σ works.

While in general, transition isomorphisms depend on the three pieces of information I,$\mathcal{U}$ and (M_i), we have constructed candidate transition isomorphisms by making a choice σ which only depends on I and $\mathcal{U}$, so we make this explicit by writing $\sigma_{I,\mathcal{U}}$.

Now, fix $\sigma_{I,\mathcal{U}}$ and let $(M_i \xrightarrow{f_i} N_i)_{i\in I}$ be an I-indexed sequence of elementary embeddings, and consider the pre-ultrafunctor diagram

$$\begin{tikzcd} X\left(\prod_{i\to\mathcal{U}} M_i\right) \arrow[r, "\Phi_{(M_i)}"] \arrow[d, "X([f_i]_{i\to\mathcal{U}})"'] & \prod_{i\to\mathcal{U}} X(M_i) \arrow[d, "{[X(f_i)]_{i\to\mathcal{U}}}"] \\ X\left(\prod_{i\to\mathcal{U}} N_i\right) \arrow[r, "\Phi_{(N_i)}"'] & \prod_{i\to\mathcal{U}} X(N_i). \end{tikzcd}$$

To show it commutes, consider an arbitrary element $[x_i]_{i\to\mathcal{U}}$ of the top left corner $X\left(\prod_{i\to\mathcal{U}} M_i\right)$. There are three cases:

1. $[x_i]_{i\to\mathcal{U}}$ is in C_{M_i}. Recall that $\Phi_{(M_i)}$ and $\Phi_{(N_i)}$ were defined to be the identities on $C_{(M_i)}, C_{N_i)}$, and that $[f_i]_{i\to\mathcal{U}}$ restricts to a map $C_{(M_i)} \to C_{(N_i)}$. Chasing $[x_i]_{i\to\mathcal{U}}$ through the diagram, we get

$$\begin{tikzcd} {[x_i]_{i\to\mathcal{U}}} \arrow[r, mapsto] \arrow[d, mapsto] & {[x_i]_{i\to\mathcal{U}}} \arrow[d, mapsto] \\ {[f_i x_i]_{i\to\mathcal{U}}} \arrow[r, mapsto] & {[f_i x_i]_{i\to\mathcal{U}}}\,. \end{tikzcd}$$

2. $[x_i]_{i\to\mathcal{U}}$ is in $A_{(M_i)}$. Recall that $[f_i]_{i\to\mathcal{U}}$ restricts to bijections $A_{(M_i)} \to A_{(N_i)}$ and $B_{(M_i)} \to B_{(N_i)}$. Chasing $[x_i]_{i\to\mathcal{U}}$ through the diagram, we get

$$\begin{tikzcd} {[x_i]_{i\to\mathcal{U}}} \arrow[r, mapsto] \arrow[d, mapsto] & {[\sigma_{I,\mathcal{U}} x_i]_{i\to\mathcal{U}}} \arrow[d, mapsto] \\ {[x_i]_{i\to\mathcal{U}}} \arrow[r, mapsto] & {[\sigma_{I,\mathcal{U}} x_i]_{i\to\mathcal{U}}}\,. \end{tikzcd}$$

3. $[x_i]_{i\to\mathcal{U}}$ is in $B_{(M_i)}$. Recall that $[f_i]_{i\to\mathcal{U}}$ restricts to bijections $A_{(M_i)} \to A_{(N_i)}$ and $B_{(M_i)} \to B_{(N_i)}$. Chasing $[x_i]_{i\to\mathcal{U}}$ through the diagram, we get

$$\begin{tikzcd} {[x_i]_{i\to\mathcal{U}}} \arrow[r, mapsto] \arrow[d, mapsto] & {[\sigma_{I,\mathcal{U}} x_i]_{i\to\mathcal{U}}} \arrow[d, mapsto] \\ {[x_i]_{i\to\mathcal{U}}} \arrow[r, mapsto] & {[\sigma_{I,\mathcal{U}} x_i]_{i\to\mathcal{U}}}\,. \end{tikzcd}$$

Therefore, after making choices of bijections $\sigma_{I,\mathcal{U}}$ for every I and $\mathcal{U}$, we obtain a transition isomorphism Φ such that (X, Φ) is a pre-ultrafunctor.

(X, Φ) is also a Δ-functor: for any ultrapower $M^{\mathcal{U}}$, recall that the subset $C_{(M_i)}$ 8.1.3 of $X\left(M^{\mathcal{U}}\right)$ contains all those elements which are constants or nonconstants that are ultraproducts of nonconstants. In particular, if $a \in M$, then $\Delta_M(a) = [a]_{i \to \mathcal{U}}$ is a constant if a is a constant or a nonconstant which is an ultraproduct of nonconstants if a is a nonconstant, so the image of $\Delta_{X(M)}$ is contained inside $C_{(M_i)} \subseteq X(M)^{\mathcal{U}}$. $X(\Delta_M)$ is just the restriction of Δ_M to $X(M)$, so the image of $X(\Delta_M)$ also lies in $C_{(M)}$ and agrees with the image of $\Delta_{X(M)}$. This means in the below diagram, the upper-left and lower-left triangles commute:

$$
\begin{tikzcd}
 & X\left(M^{\mathcal{U}}\right) \arrow[dd, bend left, "\Phi_{(M)}"] \\
X(M) \arrow[ur, "X(\Delta_M)"] \arrow[r] \arrow[dr, "\Delta_{X(M)}"'] & C_{(M)} \arrow[u] \arrow[d] \\
 & X(M)^{\mathcal{U}}.
\end{tikzcd}
$$

Furthermore, $\Phi_{(M)}$ was defined to be the identity on $C_{(M)}$, so the curved subdiagram on the right commutes. Therefore, the entire diagram commutes; in particular, the outer triangle from the definition 3.3.14 of a Δ-functor commutes, so (X, Φ) is a Δ-functor.

The theory T is countable, and by strong conceptual completeness there as many isomorphism classes of ultrafunctors as there are definable sets of T. But for any I and $\mathcal{U}$, *any* choice of a bijection $\sigma_{I,\mathcal{U}}$ worked. We will show that there are at least uncountably many isomorphism classes of Δ-functors (X, Φ) that arise from our construction. This will imply that there is some choice of Φ such that (X, Φ) is not an ultrafunctor.

Let I now be countable, and let Φ and Φ' be two different transition isomorphisms which arise from making the choices of $\sigma_{I,\mathcal{U}}$ and $\sigma'_{I,\mathcal{U}}$ during our construction. An isomorphism of pre-ultrafunctors $(X, \Phi) \to (X, \Phi')$ is an automorphism $\eta : X \to X$ such that, additionally, all diagrams of the form

$$\begin{tikzcd} X\left(\prod_{i \to \mathcal{U}} M_i\right) \arrow[r, "\Phi"] \arrow[d, "\eta_{\prod_{i \to \mathcal{U}} M_i}"'] & \prod_{i \to \mathcal{U}} X(M_i) \arrow[d, "\prod_{i \to \mathcal{U}} \eta_{M_i}"] \\ X\left(\prod_{i \to \mathcal{U}} M_i\right) \arrow[r, "\Phi'"'] & \prod_{i \to \mathcal{U}} X(M_i) \end{tikzcd}$$

commute.

By our earlier analysis 8.1.2 of the automorphisms of X, it is easy to see that when restricted to $C_{(M_i)}$, the above diagram commutes.

However, if we restrict to $A_{(M_i)} \sqcup B_{(M_i)}$, then chasing an element around the diagram

$$\begin{tikzcd} A_{(M_i)} \sqcup B_{(M_i)} \arrow[r, "\sigma_{I,\mathcal{U}}"] \arrow[d, "\eta_{\prod_{i \to \mathcal{U}} M_i}"'] & B_{(M_i)} \arrow[d, "\prod_{i \to \mathcal{U}} \eta_{M_i}"] \\ A_{(M_i)} \sqcup B_{(M_i)} \arrow[r, "\sigma'_{I,\mathcal{U}}"'] & B_{(M_i)} \end{tikzcd}$$

yields the tentative equality

$$\begin{tikzcd} {[x_i]_{i \to \mathcal{U}}} \arrow[rr, mapsto] \arrow[d, mapsto] & & \sigma_{I,\mathcal{U}}([x_i]_{i \to \mathcal{U}}) \arrow[d, mapsto] \\ {[x_i]_{i \to \mathcal{U}}} \arrow[r, mapsto] & \sigma'_{I,\mathcal{U}}([x_i]) \overset{?}{=} & \prod_{i \to \mathcal{U}} \eta_{M_i}\left(\sigma_{I,\mathcal{U}}\left([x_i]_{i \to \mathcal{U}}\right)\right) \end{tikzcd}$$

so we see that if the transition isomorphisms Φ and Φ' induced by $\sigma_{I,\mathcal{U}}$ and $\sigma'_{I,\mathcal{U}}$ are isomorphic, then there is an automorphism $\eta : X \to X$ such that $\prod_{i \to \mathcal{U}} \eta_{M_i} \circ \sigma_{I,\mathcal{U}} = \sigma'_{I,\mathcal{U}}$. Therefore, defining G to consist of all ultraproducts $\prod_{i \to \mathcal{U}} \eta_{(M_i)}$ admissible in the above diagram (so only those which restrict to a permutation on $B_{(M_i)}$), the number of isomorphism classes among the (X, Φ) is bounded from below by the number of orbits of the action by composition

$$G \rightharpoonup \text{Bijections}\left(A_{(M_i)} \sqcup B_{(M_i)}, B_{(M_i)}\right).$$

However, G can be identified with a subgroup of $\text{Sym}(\omega)^{\mathcal{U}}$. Since I was countable, $\text{Sym}(\omega)^{\mathcal{U}}$ has size $\leqslant \mathfrak{c}$ the size of the continuum.

On the other hand, the set on which G acts has the same cardinality as $|\operatorname{Sym}\left(B_{(M_i)}\right)| \geqslant 2^{\mathfrak{c}}$.

Therefore, this action has uncountably many orbits, and so there are uncountably many isomorphism classes of (X, Φ) arising from our construction. So, one of them cannot be an ultrafunctor.

We can also see that Φ can be chosen to violate a generalized diagonal embedding 3.3.13. Fix indexing sets I and J such that $|I| > |J|$, a surjection $g : I \twoheadrightarrow J$, and $\mathcal{U}$ an ultrafilter on I with $\mathcal{V}$ its pushforward $g_*\mathcal{U}$. Let $(M_j)_{j\in J}$ be a J-indexed sequence of models.

Then the associated generalized diagonal embedding $\Delta_g : \prod_{j\to\mathcal{V}} M_j \to \prod_{i\to\mathcal{U}\mathcal{V}} M_{g(i)}$ induces, informally speaking, a relationship between ultraproducts computed with respect to different indexing sets and ultrafilters: for it to be preserved, the diagram

$$\begin{tikzcd}
X\left(\prod_{j\to\mathcal{V}} M_j\right) \arrow[r, "X(\Delta_g)"] \arrow[d, "\Phi_{(M_j)}"'] & X\left(\prod_{i\to\mathcal{U}} M_{g(i)}\right) \arrow[d, "\Phi_{(M_{g(i)})}"] \\
\prod_{j\to\mathcal{V}} X(M_j) \arrow[r, "\Delta_{X(g)}"'] & \prod_{i\to\mathcal{U}} X(M_{g(i)})
\end{tikzcd}$$

must commute, for all choices of (M_j). However, our construction of Φ involved a specification of $\Phi_{(M_j)}$ based on a choice of $\sigma_{J,\mathcal{V}}$ which is independent of the choice of $\sigma'_{I,\mathcal{U}}$ used to specify $\Phi_{(M_{g(i)})}$. To make this concrete, if for a given Φ and (M_j) the diagram above happens to commute, then for any $a \in A_{(M_j)}$ in the upper-left corner which gets sent to some $b \in B_{(M_{g(i)})}$ in the lower-right corner, we can change our choice of $\Phi_{(M_j)}$ so that $\Delta_{X(g)} \circ \Phi_{(M_j)}$ sends a to a different $b' \neq b$ while keeping the rest of Φ the same, with the modified transition isomorphism Φ' still making (X, Φ') a Δ-functor.

8.1.2 The exotic pre-ultrafunctor

In the previous section, the transition isomorphisms Φ making (X, Φ) a Δ-functor were constructed to be the identity on $C_{(M)}$, and hence also restricted to the identity on the image of diagonal embeddings $\Delta_M : M \to M^{\mathcal{U}}$.

In general, $C_{(M_i)}$ splits into a disjoint union of even constants and nonconstants which are ultraproducts of nonconstants of M_i:

$$C_{(M_i)} = C^c_{(M_i)} \sqcup C^{nc}_{(M_i)}.$$

We can easily modify the construction of the transition isomorphism to *not* preserve the diagonal map, by requiring that Φ restricts to the identity only on $C^{nc}_{(M_i)}$, while on $C^c_{(M_i)}$, we now require that Φ restricts to any permutation $C_{(M_i)} \to C_{(M_i)}$, while keeping the rest of the construction the same.

Now we verify the pre-ultrafunctor condition. When we verified the pre-ultrafunctor condition during the construction of the exotic Delta-functor, we had three cases 8.1.1, according to whether an element $[x_i]_{i \to \mathcal{U}} \in X\left(\prod_{i \to \mathcal{U}} M_i\right)$ was in $A_{(M_i)}, B_{(M_i)}$ or $C_{(M_i)}$. With the new definition, the verification of the first two cases remains the same, but the case of $C_{(M_i)}$ splits into the two cases of whether $[x_i]_{i \to \mathcal{U}} \in C_{(M_i)}$ is a constant or nonconstant. If $[x_i]_{i \to \mathcal{U}}$ is a nonconstant, then since Φ still acts as the identity on $[x_i]_{i \to \mathcal{U}}$, the diagram commutes. If $[x_i]_{i \to \mathcal{U}}$ is a constant, then even if Φ restricts to a nontrivial permutation of $C_{(M_i)}$, the diagram commutes because elementary embeddings preserve constants.

However, when Φ restricts to a nontrivial permutation on the even constants, the diagonal embedding $\Delta_M : M \to M^{\mathcal{U}}$ is *not* preserved, i.e. the triangle diagram in 3.3.14 does *not* commute. For any even constant c in $X(M)$ which is not fixed by Φ (and identifying $X(M)^{\mathcal{U}}$ as a subset of $X\left(M^{\mathcal{U}}\right)$, and this as a subset of $M^{\mathcal{U}}$, $X(\Delta_M)(c) = \Delta_{X(M)}(c) = \Delta_M(c)$, but $\Phi(\Delta_{X(M)}(c)) \neq \Delta_M(c)$.

8.2 Further directions

8.2.1 Non-definable counterexamples

In this chapter, we have constructed counterexamples to Theorem 7.2.2. Thus, our counterexamples are not *a priori* counterexamples to Theorem 4.3.2. Indeed, as we pointed out in 3.3.5, our functor X *is* definable, in fact isomorphic to the functor of points of the 1-sort of T.

We therefore ask:

Question 8.2.1. What is an example of a non-definable pre-ultrafunctor?

Given the examples of exotic functors we have constructed in this section, it is natural to also ask the following questions:

Question 8.2.2. Does there exist a pre-ultrafunctor which preserves the generalized diagonal maps Δ_g, but which is not an ultrafunctor?

Question 8.2.3. Given any ultramorphism δ, does there exist a pre-ultrafunctor which preserves δ but which is not an ultrafunctor?

Question 8.2.4. Given any set of ultramorphisms S, does there exist a pre-ultrafunctor which preserves every $\delta \in S$, but fails to preserve every $\delta \notin S$?

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