Notes on Banach spaces and Hilbert spaces

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1 Summary

The category of Banach spaces and linear contractions, **Ban**, resembles that of vector spaces and linear transformations, **Vec**, except that its structure is finer; by this, I mean that it distinguishes pairs of concepts which collapse to a single concept in **Vec**.

1. **Ban** is complete and co-complete, but products and coproducts coincide only in the nullary case, and in the trivial cases which follow from that.

2. Epis and monos needn't be regular in **Ban**; one has both (regular epi/mono)- and (epi/regular mono)-factorisation systems in **Ban**, in place of a single (epi/mono)-factorisation system as in **Vec**.

3. **Ban** has two monoidal structures generalising tensor product of vector spaces; these correspond to the "tensor" and "par" of linear logic. Both have $\mathbb{C}$ as unit; hence, there is a natural transformation between them, but its components are invertible only in trivial cases.

In particular, the full subcategory of finite-dimensional Banach spaces form a $*$-autonomous category in which tensor and par differ.

Thinking of Hilbert spaces as a particular kind of Banach space, they naturally acquire the structure of a **Ban**-enriched category. But to do justice to the further (also **Ban**-enriched) "dagger" structure, one observes that there is a slightly larger class of generalised Hilbert spaces which also admit the structure of a **Ban**-enriched dagger category. The categorical yoga required to define **gHilb** requires only that **Ban** is a closed involutive monoidal category.

2 Basics about Banach spaces

A *norm* on a (real or complex) vector space V is a function $\| \ \| : V \to [0, \infty)$ satisfying

1. $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$

2. $\|\lambda \cdot \gamma\| = |\lambda| \cdot \|\gamma\|$

3. $\|\alpha\| = 0 \Rightarrow \alpha = 0$

If $\sum \|\alpha_k\| < \infty$, then the partial sums $\sum_{k<n} \alpha_k$ form a Cauchy sequence. Hence, in the presence of Cauchy-completeness, one can define $\sum \alpha_k$ to be the limit of $\sum_{k<n} \alpha_k$ whenever $\sum \|\alpha_k\| < \infty$. Such series are naturally called *norm-convergent*.

A *Banach space* P is a complex vector space p together with a norm $\| \ \|_P$ for which p is Cauchy-complete. The set of vectors of norm ≤ 1 is called the *unit ball* of P and denoted $\mathcal{U}(P)$.

$$\mathcal{U}(P) = \{\alpha \in p \mid \ \|\alpha\|_P \leq 1\} \tag{1}$$

A *linear contraction* $P \to Q$ is a linear transformation $\omega : p \to q$ satisfying $\|\omega(\alpha)\|_Q \leq \|\alpha\|_P$; equivalently, one which restricts to a map $\mathcal{U}(P) \to \mathcal{U}(Q)$. More generally, a *multilinear contraction* $P_1 \times \ldots \times P_n \to Q$

is a multilinear transformation $\psi : p_1 \times \ldots \times p_n \to q$ satisfying $\|\psi(\alpha_1, \ldots, \alpha_n)\|_Q \leq \|\alpha_1\|_{P_1} \cdots \|\alpha_n\|_{P_n}$; equivalently, one which restricts to a map $\mathcal{U}(P_1) \times \cdots \times \mathcal{U}(P_n) \to \mathcal{U}(Q)$.

The symmetric multicategory of Banach spaces and multilinear contractions, $\mathbf{Ban}_{multi}$, happens to be representable; that is, for every n-tuple of Banach spaces $(P_1, \ldots, P_n)$, there exists a universal multilinear contraction $\otimes : P_1 \times \ldots \times P_n \to P_1 \boxtimes \ldots \boxtimes P_n$. Explicitly, for $n > 0$, $P_1 \boxtimes \ldots \boxtimes P_n$ is the Cauchy completion of $p_1 \otimes \ldots \otimes p_n$ with respect to the following norm.

$$\|\beta\|_{\boxtimes} = \inf \left\{ \sum_{j<k} \|\alpha_{1,j}\|_{P_1} \cdots \|\alpha_{n,j}\|_{P_n} \;\middle|\; \beta = \sum_{j<k} \alpha_{1,j} \otimes \cdots \otimes \alpha_{n,j} \right\} \tag{2}$$

(In the nullary case, we take `"1"` $: 1 \to \mathbb{C}$.) Thus the (mere) category of Banach spaces and linear contractions, **Ban**, admits a symmetric monoidal structure defined on objects by the binary and nullary cases of $\boxtimes$.

A bilinear contraction $\psi : P \times Q \to R$ can be recast as a bilinear transformation $\psi : p \times q \to r$ satisfying

$$\sup \{\|\psi(\alpha, \beta)\|_R \mid \alpha \in \mathcal{U}(P)\} \leq \|\beta\|_Q \tag{3}$$

—hence, the Curry of ψ (*qua* linear transformation $q \to p \multimap r$) factors through $s := \{\omega \in p \multimap r \mid \|\omega\|_S < \infty\}$, where

$$\|\omega\|_S := \sup \{\|\omega(\alpha)\|_R \mid \alpha \in \mathcal{U}(P)\}, \tag{4}$$

and is furthermore contractive with respect to $\| \, \|_Q$ and $\| \, \|_S$. It is easily verified that s and $\| \, \|_S$ form a Banach space, henceforth denoted $P \multimap R$ or $R \multimapinv P$. Thus the symmetric monoidal category $(\mathbf{Ban}, \boxtimes, \mathbb{C})$ is closed.

It happens that a linear transformation $\omega : p \to r$ satisfies $\omega \in P \multimap R$ (*i.e.*, $\|\omega\|_{P \multimap R} < \infty$) if and only if it is continuous with respect to the topologies induced by $\| \, \|_P$ and $\| \, \|_R$.

3 More about Banach spaces

Clearly, $\mathcal{U}$ defines a functor $\mathbf{Ban} \to \mathbf{Set}$; moreover, the universal bilinear contraction $\otimes : P \times Q \to P \boxtimes Q$ restricts to a map $\mathcal{U}(P) \times \mathcal{U}(Q) \to \mathcal{U}(P \boxtimes Q)$, which, by abuse of notation, we also denote $\otimes$. Then $(\mathcal{U}, \otimes, 1)$ defines a monoidal functor $(\mathbf{Ban}, \boxtimes, \mathbb{C}) \to (\mathbf{Set}, \times, 1)$. This has a monoidal left adjoint whose functor part is denoted ℓ^1; explicitly, $\ell^1(J) = \{\alpha : J \to \mathbb{C} \mid \|\alpha\|_{\ell^1} < \infty\}$, where $\|\alpha\|_{\ell^1} = \sum |\alpha_j|$.

This adjunction may be monoidal, but it is not monadic. Nevertheless, the comparison functor $\mathbf{Ban} \to \mathbf{Alg}(\mathcal{U}\ell^1)$ is fully faithful and reflective; moreover, $\mathcal{U}\ell^1$ is $\aleph_1$-ary. We can conclude that **Ban** is locally $\aleph_1$-presentable—therefore, in particular, complete, co-complete, and not self-dual. However, the full subcategory $\mathbf{Ban}_{fd}$ of finite-dimensional Banach spaces is $*$-autonomous.

The dual tensor product ("par"), defined on $\mathbf{Ban}_{fd}$ by

$$P \bowtie Q := ({}^*Q \boxtimes {}^*P)^* \cong P \multimapinv {}^*Q,$$

extends to the whole of **Ban**. Explicitly, for arbitrary Banach spaces P and Q, $P \bowtie Q$ is the Cauchy completion of $p \otimes q$ with respect to the following norm.

$$\|\zeta\|_{\bowtie} = \sup \{|(\omega \otimes \psi)(\zeta)| \mid \omega \in \mathcal{U}(P \multimap \mathbb{C}), \psi \in \mathcal{U}(Q \multimap \mathbb{C})\} \tag{5}$$

Whenever $\zeta \in p \otimes q$ can be written in the form $\sum_{j<k} \alpha_j \otimes \beta_j$, we have

$$\begin{aligned} \|\zeta\|_{\bowtie} &= \sup \left\{ \left| (\omega \otimes \psi)(\sum_{j<k} \alpha_j \otimes \beta_j) \right| \;\middle|\; \omega \in \mathcal{U}(P \multimap \mathbb{C}), \psi \in \mathcal{U}(Q \multimap \mathbb{C}) \right\} \\ &= \sup \left\{ \left| \sum_{j<k} \omega(\alpha_j) \cdot \psi(\beta_j) \right| \;\middle|\; \omega \in \mathcal{U}(P \multimap \mathbb{C}), \psi \in \mathcal{U}(Q \multimap \mathbb{C}) \right\} \end{aligned}$$

$$
\begin{aligned}
&\leq \sup\left\{\sum_{j<k}|\omega(\alpha_j)|\cdot|\psi(\beta_j)| \;\middle|\; \omega\in\mathcal{U}(P\multimap\mathbb{C}),\psi\in\mathcal{U}(Q\multimap\mathbb{C})\right\}\\
&\leq \sum_{j<k}\|\alpha_j\|\cdot\|\beta_j\|
\end{aligned}
$$

—hence $\|\zeta\|_{\bowtie} \leq \|\zeta\|_{\boxtimes} = \inf\left\{\sum_{j<k}\|\alpha_j\|\cdot\|\beta_j\| \;\middle|\; \zeta = \sum_{j<k}\alpha_j\otimes\beta_j\right\}$. Thus the identity on $p\otimes q$ extends to a linear contraction $\nu : P \boxtimes Q \to P \bowtie Q$. Curiously, ν need not be injective: the "nuclear tensor product" of Banach spaces is defined by $P \boxtimes_{nuc} Q = (P\boxtimes Q)/\ker\nu$.

Similarly, for arbitrary Banach spaces P, Q, R, the associativity isomorphism $p\otimes(q\otimes r)\to(p\otimes q)\otimes r$ extends to a linear contraction

$$P\boxtimes(Q\bowtie R)\to(P\boxtimes Q)\bowtie R$$

generalising the linear distributions which come for free in the finite-dimensional case.

On the subject of which, a linear contraction $\omega : P \to Q$ is: mono, if it is injective; regular mono, if it preserves norm; epi, if its range is dense; regular epi, if $\|\beta\|_Q = \inf\{\|\alpha\|_P \mid \omega(\alpha)=\beta\}$. So the epi/regular-mono factorisation of ω is given by the closure of its range, with the restriction of $\|\;\|_Q$; the regular-epi/mono factorisation of ω is given by equipping $P/\ker\omega$ with the appropriate norm. It is important fact that $\mathbb{C}$ is a regular-injective object in **Ban**—that is, injective wrt regular monos. From this, it follows that the canonical maps $P\to({}^*P)^*$ are regular mono. We recall that an object of a closed monoidal category is called *reflexive* if that map happens to be iso; thus to show that a Banach space P is reflexive, it suffices to show that the range of $P\to({}^*P)^*$ is dense.

Finally, the conjugate of a complex vector space V, denoted $\overline{V}$, is defined by changing the scalar multiplication of V and leaving all else the same; similarly also the conjugate of a Banach space; $V\mapsto\overline{V}$ underlies an endofunctor of **Vec**, and $P\mapsto\overline{P}$ underlies an endofunctor of **Ban**, in each case defined on arrows by $\omega\mapsto\omega$. There is a natural isomorphism $\overline{\mathrm{span}(J)}\to\mathrm{span}(J)$, where span denotes the left adjoint of the forgetful functor $\mathbf{Vec}\to\mathbf{Set}$; similarly, there is a natural isomorphism $\overline{\ell^1(J)}\to\ell^1(J)$. Since (arguably) every vector space is isomorphic to one of the form $\mathrm{span}(J)$, one can conclude that there is an isomorphism $\overline{V}\xrightarrow{\sim}V$ for every vector space V, hence also an isomorphism $\overline{P}\xrightarrow{\sim}P$ for every Banach space P. Nevertheless, there is no natural isomorphism $\overline{V}\xrightarrow{\sim}V$; nor a natural isomorphism $\overline{P}\xrightarrow{\sim}P$. But there is a coherent natural isomorphism $\overline{V\otimes W}\xrightarrow{\sim}\overline{W}\otimes\overline{V}$ defined by $\alpha\otimes\beta\mapsto\beta\otimes\alpha$; this induces to coherent natural isomorphisms $\overline{P\boxtimes Q}\xrightarrow{\sim}\overline{Q}\boxtimes\overline{P}$ and $\overline{P\bowtie Q}\xrightarrow{\sim}\overline{Q}\bowtie\overline{P}$. Thus $(\mathbf{Vec},\otimes,\mathbb{C},\overline{(\;)})$ is a semitrivial example of an involutive monoidal category, and $(\mathbf{Ban},\boxtimes,\mathbb{C},\bowtie,\mathbb{C},\overline{(\;)})$ that of an involutive linear-distributive category.

4 Basics about Hilbert spaces

A *Hermitian form* on a vector space V is a function $\langle -,-\rangle : V\times V\to\mathbb{C}$ satisfying

1. $\langle\alpha,\beta+\gamma\rangle = \langle\alpha,\beta\rangle+\langle\alpha,\gamma\rangle$
2. $\langle\alpha,\lambda\cdot\beta\rangle = \lambda\cdot\langle\alpha,\beta\rangle$
3. $\overline{\langle\alpha,\beta\rangle} = \langle\beta,\alpha\rangle$

A Hermitian form is called: *definite*, if $\langle\alpha,\alpha\rangle=0\Rightarrow\alpha=0$; *positive*, if $\langle\alpha,\alpha\rangle\geq 0$. An *inner product* is a Hermitian form which is both positive and definite. The *induced norm* of an inner product is $\|\alpha\|=\sqrt{\langle\alpha,\alpha\rangle}$. It is an easy theorem that

$$|\langle\alpha,\beta\rangle|\leq\|\alpha\|\cdot\|\beta\| \tag{6}$$

for any inner product (and its induced norm) on any vector space.

A *Hilbert space* $\mathbf{h}$ is conventionally defined as a vector space h together with an inner product $\langle -,-\rangle_{\mathbf{h}}$ such that h is Cauchy-complete in the induced norm of $\langle -,-\rangle_{\mathbf{h}}$. We denote the induced norm of $\langle -,-\rangle_{\mathbf{h}}$ by $\|\;\|_H$ (not $\|\;\|_{\mathbf{h}}$); the Banach space comprising h and $\|\;\|_H$ will be called the *induced Banach space* of $\mathbf{h}$, and it will be denoted H. The map $\mathbf{h} \mapsto H$, together with the canonical self-enrichment of **Ban**, allows us to define a $(\mathbf{Ban}, \boxtimes, \mathbb{C})$-enriched category of Hilbert spaces, **Hilb**. Explicitly, $\mathbf{Hilb}(\mathbf{h}, \mathbf{k}) = K \multimapinv H$.

It is commonplace to observe that an inner product on h may be construed as a bilinear map $\overline{h} \times h \to \mathbb{C}$; hence also as a linear map $\overline{h} \otimes h \to \mathbb{C}$. However, the Cauchy-Schwarz inequality (6 above) entails that it also defines a bilinear contraction $\overline{H} \times H \to \mathbb{C}$; hence also a linear contraction $\overline{H} \boxtimes H \to \mathbb{C}$. The latter can be Curryed into a linear contraction $\upsilon : \overline{H} \to {}^*H$, and, according to the Riesz Representation Theorem, this Curryed form is surjective. Now it is easy to show that $\overline{H} \to {}^*H$ preserves norm. In fact, for an arbitrary contraction of the form $\langle -, - \rangle : K \boxtimes H \to \mathbb{C}$, the assertion that its Curry $K \to {}^*H$ preserve norm is equivalent to the assertion

$$(\forall \alpha \in K)(\exists \beta \in H)\, |\langle \alpha, \beta \rangle| = \|\alpha\| \cdot \|\beta\| \tag{7}$$

—so, in the case at hand, one can simply choose $\beta = \alpha$. Thus a Hilbert space $\mathbf{h}$ may be regarded as a Banach space together with a particular sort of isomorphism $\overline{H} \to {}^*H$.

Specifically, the Hermitianness axiom $(\overline{\langle \alpha, \beta \rangle} = \langle \beta, \alpha \rangle)$, when expressed diagrammatically,

$$\begin{array}{ccc}
\overline{\overline{H} \boxtimes H} & \xrightarrow{\langle -,- \rangle} & \overline{\mathbb{C}} \\
\chi \uparrow & & \\
\overline{H} \boxtimes \overline{\overline{H}} & & \downarrow \overline{(\ )} \\
= \downarrow & & \\
\overline{H} \boxtimes H & \xrightarrow{\langle -,- \rangle} & \mathbb{C}
\end{array}$$

(where χ denotes the isomorphism given by $\alpha \otimes \beta \mapsto \beta \otimes \alpha$ discussed above) can be Curryed as follows.

$$\begin{array}{ccccc}
\overline{\overline{H}} & \xrightarrow{\upsilon} & {}^*\overline{H} & \xrightarrow{\sim} & \overline{H}^* \\
\| & & & & \uparrow \upsilon^* \\
H & & \longrightarrow & & ({}^*H)^*
\end{array}$$

So, in particular, a Hilbert space is reflexive as a Banach space.

These facts are what allows us to define an (enriched) dagger structure on **Hilb**. Briefly, $(\ )^\dagger : \overline{\mathbf{Hilb}(\mathbf{h}, \mathbf{k})} \to \mathbf{Hilb}(\mathbf{k}, \mathbf{h})$ is defined by

$$\overline{K \multimapinv H} \xrightarrow{\sim} \overline{H} \multimap \overline{K} \xrightarrow{\sim} {}^*H \multimap {}^*K \xrightarrow{\sim} H \multimapinv K$$

where the last isomorphism is the inverse of the canonical map $H \multimapinv K \to {}^*H \multimap {}^*K$; equivalently, the canonical map ${}^*H \multimap {}^*K \to ({}^*H)^* \multimapinv ({}^*K)^*$, composed with the isos $H \xrightarrow{\sim} ({}^*H)^*$ and $K \xrightarrow{\sim} ({}^*K)^*$.

5 Generalised Hilbert spaces

Let us define a *contractive Hermitian form* on a Banach space H to be one satisfying the Cauchy-Schwarz inequality (6)—equivalently, a linear contraction $\langle -, - \rangle : \overline{H} \boxtimes H \to \mathbb{C}$ satisfying the Hermitianness diagram above. Let us say that a contractive Hermitian form is *weakly definite* it satisfies (7) above—equivalently, if its Curry $\upsilon : \overline{H} \to {}^*H$ is invertible. Finally, we define a *generalised Hilbert space* $\mathbf{h}$ to comprise a Banach space H together with a weakly definite contractive Hermitian form $\langle -, - \rangle_{\mathbf{h}}$. Then there is a **Ban**-enriched dagger category **gHilb** of generalised Hilbert spaces with $\mathbf{gHilb}(\mathbf{h}, \mathbf{k}) = K \multimapinv H$, and dagger map $\overline{\mathbf{gHilb}(\mathbf{h}, \mathbf{k})} \to \mathbf{gHilb}(\mathbf{k}, \mathbf{h})$ defined exactly as above.

Given a Hilbert space $\mathbf{h}$ and a self-adjoint unitary σ (*i.e.*, an arrow $H \to H$ with $\sigma = \sigma^\dagger = \sigma^{-1}$), one can define a generalised Hilbert space $\mathbf{h}^\sigma$ having the same underlying Banach space H, but the "deviant" inner product $\langle \alpha, \beta \rangle_{\mathbf{h}^\sigma} = \langle \alpha, \sigma\beta \rangle_{\mathbf{h}}$. Given two such generalised Hilbert spaces, $\mathbf{h}^\sigma$ and $\mathbf{k}^\tau$, and an $\omega \in K \multimapinv H$, we have

$$\begin{aligned} \langle \alpha, \omega\beta \rangle_{\mathbf{k}^\tau} &= \langle \alpha, \tau\omega\beta \rangle_{\mathbf{k}} \\ &= \left\langle \omega^\dagger \tau\alpha, \beta \right\rangle_{\mathbf{h}} \\ &= \left\langle \sigma\omega^\dagger \tau\alpha, \sigma\beta \right\rangle_{\mathbf{h}} \\ &= \left\langle \sigma\omega^\dagger \tau\alpha, \beta \right\rangle_{\mathbf{h}^\sigma} \end{aligned}$$

—hence the "deviant" dagger (which we denote ‡) $\overline{\mathbf{gHilb}(\mathbf{h}^\sigma, \mathbf{k}^\tau)} \to \mathbf{gHilb}(\mathbf{k}^\tau, \mathbf{h}^\sigma)$ relates to the conventional dagger $\overline{\mathbf{Hilb}(\mathbf{h}, \mathbf{k})} \to \mathbf{Hilb}(\mathbf{k}, \mathbf{h})$ of Hilbert spaces via $\omega^\ddagger = \sigma\omega^\dagger\tau$.

If I understand correctly, every generalised Hilbert space is equivalent (in a dagger category theoretic sense) to one of the form $\mathbf{h}^\sigma$. (I just need to find an appropriate citation.) Thus, in particular, the underlying categories of **Hilb** and **gHilb** are equivalent. But the underlying dagger categories of **Hilb** and **gHilb** are certainly not equivalent.

Generalised Hilbert spaces are not much of a generalisation: they arose purely because I have never quite figured out how to express positivity in a sufficiently categorical manner, and because they suffice for my purposes. Or, at least, they have sufficed for my purposes so far.