# On open locales and inner products

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## Abstract

Given an open locale $E$, the operation $[\alpha, \beta] = \exists_!(\alpha \wedge \beta)$ defines a sort of "inner product" on the underlying frame of $E$. In this talk, we explore fruitful analogies between the theory of inner product spaces and open locales, touching on the theory of uniform and metric locales, as well as modal logic and orthomodular lattices.

## Review: uniform continuity

Given $(E, \varphi_E)$, $(F, \varphi_F)$ be (generalised) metric spaces, $\omega\colon E \to F$ is uniformly continuous if

$$(\forall \varepsilon\colon \mathbb{Q}_+)(\exists \delta\colon \mathbb{Q}_+)(\forall \sigma\colon E)(\forall \tau\colon E)(\varphi_E(\sigma,\tau) < \delta \Rightarrow \varphi_F(\omega\sigma,\omega\tau) < \varepsilon).$$

Setting $N_\varepsilon = \{(\sigma,\tau) \mid \varphi(\sigma,\tau) < \varepsilon\}$, this means

$$(\forall \varepsilon\colon \mathbb{Q}_+)(\exists \delta\colon \mathbb{Q}_+)(\forall \pi\colon E \times E)(\pi \in N_\delta \Rightarrow (\omega \times \omega)(\pi) \in N_\varepsilon)$$

—equivalently, $(\forall \varepsilon\colon \mathbb{Q}_+)(\exists \delta\colon \mathbb{Q}_+)(N_\delta \subseteq (\omega \times \omega)^* N_\varepsilon)$.

$N_\varepsilon$'s are called *basic entourages*; supersets of these are called *entourages*. Then $\omega$ is uniformly continuous iff $(\omega \times \omega)^*$ preserves entourages.

## Example: left and right uniformities

Given a topological group $G$, and $1 \in U \in \mathcal{O}(G)$, let

$$L_U = \left\{(\alpha, \beta) \colon G \times G \;\middle|\; \alpha^{-1}\beta \in U\right\}$$

then

$$\{N \subseteq G \times G \mid (\exists U \colon \mathcal{O}(G))(1 \in U \text{ and } L_U \subseteq N)\}$$

defines the *left uniformity* of $G$.

Similarly,

$$\{N \subseteq G \times G \mid (\exists U \colon \mathcal{O}(G))(1 \in U \text{ and } R_U \subseteq N)\}$$

defines the *right uniformity* of $G$, where

$$R_U = \left\{(\alpha, \beta) \colon G \times G \;\middle|\; \alpha\beta^{-1} \in U\right\}.$$

## Naïve idea: entourages as sup-homomorphisms

An entourage is an element of $\mathcal{P}(E \times E) \cong \mathcal{P}(E) \multimapinv \mathcal{P}(E)$.

Replace each relation $N \subseteq E \times E$ with the corresponding sup-homomorphism $\Diamond \colon \mathcal{P}(E) \to \mathcal{P}(E)$.

In the metric case,

$$\Diamond_\varepsilon(A) = \{\tau \colon E \mid (\exists \sigma \colon E)(\sigma \in A \text{ and } \varphi_E(\sigma, \tau) < \varepsilon)\}$$

—so $\Diamond_\varepsilon$ "fattens" a set $A$ by $\varepsilon$.

Being a sup-homomorphism, $\Diamond$ has a right adjoint $\Box$, given by

$$\Box_\varepsilon(B) = \{\sigma \colon E \mid (\forall \tau \colon E)(\varphi_E(\sigma, \tau) < \varepsilon \Rightarrow \tau \in B)\}$$

—so $\Box_\varepsilon$ "shrinks" a set $B$ by $\varepsilon$.

Note $\bigvee_{\varepsilon \colon \mathbb{Q}_+} \Box_\varepsilon(B)$ is the interior of $B$ in the metric topology.

## Overview: analysis and topology

Classically, downward arrows are derivations.

$$
\begin{array}{ccc}
\text{inner products} & & \\
\Downarrow & & \\
\text{norms} & & \\
\Downarrow & & \\
\text{metrics} & \subseteq & \text{quasi-metrics} \\
\Downarrow & & \Downarrow \\
\text{uniformities} & \subseteq & \text{quasi-uniformities} \\
\Downarrow & & \Downarrow \\
\text{completely regular topologies} & \subseteq & \text{topologies}
\end{array}
$$

Constructively, seems better to make them forgetful.

## Motivation: (NCG via TT) via open groupoids

$$\left.\begin{array}{c}\text{locally compact}\\ \text{groupoids}\\ +\text{Haar system}\end{array}\right\} \to \text{C*-algebras} \to \left\{\begin{array}{c}\text{categories of}\\ \text{Hilbert C*-modules}\end{array}\right.$$

In 1-object case, Haar system comes "for free".

In general, it doesn't; in particular, if $\mathcal{G} = (G_1 \rightrightarrows G_0)$ admits a Haar system, then it is *open*.

## Review: Hermitian objects in an IMC

A *Hermitian object* in an IMC $(\mathcal{V}, \otimes, \overline{(\ )}, I)$ is an object $H$ together with a map $\gamma\colon \overline{H} \otimes H \to I$ satisfying

$$\begin{array}{ccc}
\overline{\overline{H} \otimes H} & \xrightarrow{\quad\overline{\gamma}\quad} & \overline{I} \\
\sim\uparrow & & \\
\overline{H} \otimes \overline{\overline{H}} & & \Big\downarrow (\ )^{\dagger} \\
\sim\downarrow & & \\
\overline{H} \otimes H & \xrightarrow{\quad\gamma\quad} & I
\end{array}$$

A Hermitian object $(H, \gamma)$ is called *weakly definite* if:

1) $H$ is exponentiable;

2) $\gamma$'s transpose $ket\colon \overline{H} \to I \multimap H$ is invertible.

## Review: dagger

In a closed IMC, the Hermitianness axiom is equivalent to

$$
\begin{array}{ccccc}
\overline{\overline{H}} & \xrightarrow{\overline{ket}} & \overline{I \multimapinv H} & \xrightarrow{\sim} & \overline{H} \multimap \overline{I} \\
{\scriptstyle\sim}\big\downarrow & & & & \big\downarrow{\scriptstyle \overline{H} \multimap (\ )^{\dagger}} \\
H & \longrightarrow & (I \multimapinv H) \multimap I & \xrightarrow[ket \multimap I]{} & \overline{H} \multimap I
\end{array}
$$

—so, in particular, a weakly definite Hermitian object is *reflexive*; from this fact, we can derive the "adjoint" of a map between weakly definite Hermitian objects.

$$
\overline{K \multimapinv H} \xrightarrow{(\ )^{\dagger}} H \multimapinv K
$$

In this way, weakly definite Hermitian objects from a *dagger* $(\mathcal{V}, \otimes, \overline{(\ )}, I)$*-category*.

## Aside: adjointable maps

A Hermitian object is a species of Chu space—one which is almost equal on the nose to its Chu dual; for arbitrary Hermitian objects in an IMC with pullbacks, the space of Chu morphisms

$$
\begin{array}{ccc}
Chu((H,\overline{H},\gamma),(K,\overline{K},\kappa)) & \longrightarrow & K \circ\!\!- H \\
 & & \big\downarrow bra_K \circ\!\!- H \\
\big\downarrow & & (\overline{K} \multimap I) \circ\!\!- H \\
 & & \big\downarrow \sim \\
\overline{K} \multimap \overline{H} & \xrightarrow{\overline{K} \multimap ket_H} & \overline{K} \multimap (I \circ\!\!- H)
\end{array}
$$

admits an analogous dagger operation.

## Review: open locales

A map of locales $\omega: E \to F$ is *open* if $\omega^*: \mathcal{O}(F) \to \mathcal{O}(E)$ is a *complete Heyting algebra homomorphism*—*i.e.*, preserves all meets and joins, and also $\Rightarrow$.

(Note that this implies $\omega^*$ has a left adjoint $\exists_\omega$ as well as the usual right adjoint $\forall_\omega$.)

Classically, every map of the form $!: E \to 1$ is open; but this needn't be true in an arbitrary topos $\mathcal{T}$. So we call a $\mathcal{T}$-locale *open* (or, *locally positive*) if $!: E \to 1$ is open.

## Intuition: positivity

Any map $X \to \Omega$ is a predicate; given an open locale $E$,

$$\begin{array}{ccc} \mathcal{O}(E) & \xrightarrow{\exists_!} & \Omega \\ & {\scriptstyle \llcorner \bot \lrcorner} \searrow & \downarrow {\scriptstyle \neg} \\ & & \Omega^{op} \end{array}$$

where $\llcorner \bot \lrcorner(\alpha)$ is the truth value of $\alpha \leq \bot$; equivalently, $\alpha = \bot$.

So $\exists_!$ is a predicate whose negation is emptiness; we call it *positivity*. (Non-emptiness=non-non-positivity!)

## Hermitian sup-lattices

Given a topos $\mathcal{T}$, let $(\mathbf{Sup}(\mathcal{T}); \otimes, \Omega)$ denote the (closed) symmetric monoidal category of complete lattices and supremum-preserving maps in $\mathcal{T}$.

Given an open $\mathcal{T}$-locale $E$, its underlying frame $\mathcal{O}(E)$ together with

$$\mathcal{O}(E) \otimes \mathcal{O}(E) \xrightarrow{\ \wedge\ } \mathcal{O}(E) \xrightarrow{\ \exists_!\ } \Omega \qquad \gamma : \mathcal{O}(E) \otimes \mathcal{O}(E) \to \Omega$$

form a Hermitian object in $(\mathbf{Sup}(\mathcal{T}); \otimes, \Omega)$—regarded as an IMC by setting $\overline{\ell} = \ell$.

## Weakly definite Hermitian sup-lattices

Classically, $(\mathcal{O}(E), \gamma)$ is weakly definite iff $\mathcal{O}(E)$ is boolean; but this is not true in general.

In general, we have

$$\begin{array}{ccccc} \mathcal{O}(E) \owedge \mathcal{O}(E) & \xrightarrow{\quad \wedge \quad} & \mathcal{O}(E) & \xrightarrow{\quad \exists_! \quad} & \Omega \\ & & & \searrow_{\llcorner \perp \lrcorner} & \downarrow{\neg} \\ & & & & \Omega^{op} \end{array}$$

which Curries into

$$\begin{array}{ccc} \mathcal{O}(E) & \xrightarrow{\quad ket \quad} & \Omega \mathrel{\circ\!\!-} \mathcal{O}(E) \\ \neg \downarrow & & \downarrow \neg \mathrel{\circ\!\!-} \mathcal{O}(E) \\ \mathcal{O}(E)^{op} & \xrightarrow{\quad \sim \quad} & \Omega^{op} \mathrel{\circ\!\!-} \mathcal{O}(E). \end{array}$$