What is a morphism of Frobenius functors?

(also: what are *lax* Frobenius functors?)

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Review: a *ldc* is a category with two monoidal structures

($\owedge$ and $\ovee$) or ($\overset{\frown}{\times}$ and $\underset{\smile}{\times}$) or ($\otimes$ and $\parr$) or . . .

linked by nats

$$p \owedge (q \ovee r) \longrightarrow (p \owedge q) \ovee r \qquad\qquad p \overset{\frown}{\times} (q \underset{\smile}{\times} r) \longrightarrow (p \overset{\frown}{\times} q) \underset{\smile}{\times} r$$

$$(q \ovee r) \owedge s \longrightarrow q \ovee (r \owedge s) \qquad\qquad (q \underset{\smile}{\times} r) \overset{\frown}{\times} s \longrightarrow q \underset{\smile}{\times} (r \overset{\frown}{\times} s)$$

Mnemonic Device: "linear discipline"

$p \wedge (q \vee r) \quad = \quad (p \wedge q) \vee (p \wedge r)$ —bad! (repetition of p)
$p \wedge (q \vee r) \quad \leqslant \quad (p \wedge q) \vee r$ —good! (no repetition of p)

Naïve question: is this *only* a mnemonic device?

Review: a (unital) *quantale* is a complete lattice with an associative (and unital) operation $\&$ which distributes joins:

$$\alpha \,\&\, (\bigvee_{j \in I} \beta_j) = \bigvee_{j \in I} (\alpha \,\&\, \beta_j) \qquad (\bigvee_{j \in I} \beta_j) \,\&\, \zeta = \bigvee_{j \in I} (\beta_j \,\&\, \zeta)$$

There exists a monoidal category $(\mathbf{Sup}, \otimes, 2)$ whose monoids are the same thing as unital quantales.

Definition: a (unital) *ldq* is a (unital) quantale plus a second associative (and unital) operation $⅋$ which distributes meets

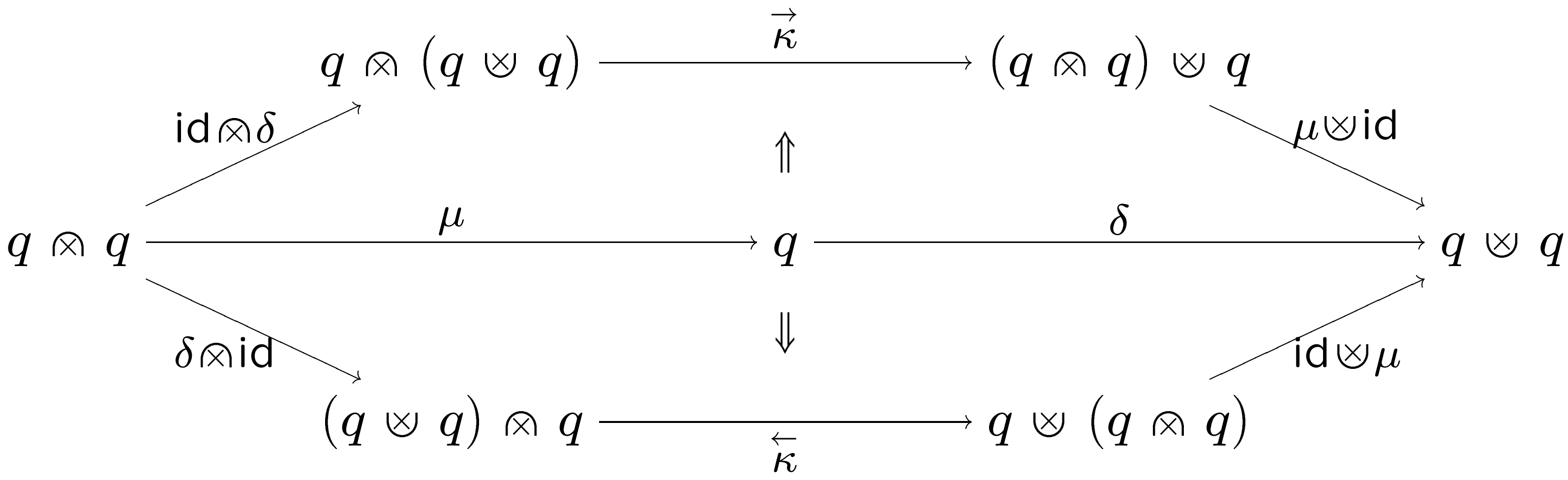
$$\alpha \,⅋\, (\bigwedge_{j \in I} \beta_j) = \bigwedge_{j \in I} (\alpha \,⅋\, \beta_j) \qquad (\bigwedge_{j \in I} \beta_j) \,⅋\, \zeta = \bigwedge_{j \in I} (\beta_j \,⅋\, \zeta)$$

and such that

$$\begin{aligned} \alpha \,\&\, (\beta \,⅋\, \zeta) &\leqslant (\alpha \,\&\, \beta) \,⅋\, \zeta \\ (\beta \,⅋\, \zeta) \,\&\, \vartheta &\leqslant \beta \,⅋\, (\zeta \,\&\, \vartheta) \end{aligned}$$

hold.

Theorem: a (unital) ldq is the same thing as a semigroup (resp., monoid) in $(\mathbf{Sup}, \otimes, 2)$ and a cosemigroup (resp., comonoid) in $(\mathbf{Sup}, \bar{\otimes}, 2^{\mathsf{op}})$ which share the same underlying object, q, together with two 2-cells as below.



Definition: a *Frobenius quantale* (Frq) is a unital ldq where the two 2-cells are identities.

[Note: this does *not* entail equalities $\alpha \,\&\, (\beta \parr \zeta) = (\alpha \,\&\, \beta) \parr \zeta$, $(\beta \parr \zeta) \,\&\, \vartheta = \beta \parr (\zeta \,\&\, \vartheta)$.]

Theorem: a Frq amounts to a unital ldq *with duals*; that is, a quantale which is also $*$-autonomous.

Remark: there exists a notion of *Frobenius functor* (Frf) between arbitrary ldcs such that a Frf $\mathbb{1} \to (\mathbf{Sup}, \otimes, \mathbf{2}, \oplus, \mathbf{2}^{\mathsf{op}})$ is, tautologously, a Frq. (More later!)

Theorem: let s, t be (unital) ldqs, $t \xrightarrow{\psi} s$ an arrow in $\mathbf{Sup}$, and $\psi^\sharp$ its right adjoint. Then ψ is a morphism of cosemigroups (resp., comonoids) iff $\psi^\sharp$ preserves $\otimes$ (and its unit).

Definition: a morphism of (unital) ldqs is a $\mathbf{Sup}$-adjunction

$$s \underset{\psi}{\overset{\omega}{\rightleftarrows}} t$$

such that ω is a morphism of semigroups (resp., monoids) and ψ a morphism of cosemigroups (resp., comonoids).

A morphism of Frqs "should be" a morphism of unital ldqs.

Question: can we find an equivalent definition (of morphism of Frqs) which does not refer to the 2-structure of $\mathbf{Sup}$?

Review: a *linear functor* (lf) between ldcs

$$(\mathcal{J}, \otimes, e, \oplus, d) \xrightarrow{\quad T \quad} (\mathcal{K}, \otimes, e, \oplus, d)$$

consists of: a monoidal functor $(\mathcal{J}, \otimes, e) \xrightarrow{\forall_T} (\mathcal{K}, \otimes, e)$, a comonoidal functor $(\mathcal{J}, \oplus, d) \xrightarrow{\exists_T} (\mathcal{K}, \oplus, d)$, actions of $\forall_T$ on $\exists_T$

$$\forall_T(p) \otimes \exists_T(q) \longrightarrow \exists_T(p \otimes q) \longleftarrow \exists_T(p) \otimes \forall_T(q)$$

and coactions of $\exists_T$ on $\forall_T$

$$\exists_T(p) \oplus \forall_T(q) \longleftarrow \forall_T(p \oplus q) \longrightarrow \forall_T(p) \oplus \exists_T(q).$$

satisfying a number of compatibility axioms.

A Frf is a lf with $\forall_T = \exists_T$ *and trivial actions and coactions*.

Review: a morphism of lfs $S \xrightarrow{\vartheta} T$ is a monoidal nat $\forall_S \xrightarrow{\forall_\vartheta} \forall_T$ together with a comonoidal nat $\exists_T \xrightarrow{\exists_\vartheta} \exists_S$ which further satisfy

$$\begin{array}{ccccc}
\forall_S(p) \uplus \exists_S(q) & \xrightarrow{\forall_\vartheta(p)\uplus\mathsf{id}} & \forall_T(p) \uplus \exists_S(q) & \xleftarrow{\mathsf{id}\uplus\exists_\vartheta(q)} & \forall_T(p) \uplus \exists_T(q) \\
\uparrow & & & & \uparrow \\
\forall_S(p \uplus q) & & \xrightarrow{\qquad\forall_\vartheta(p\uplus q)\qquad} & & \forall_T(p \uplus q) \\
\downarrow & & & & \downarrow \\
\exists_S(p) \uplus \forall_S(q) & \xrightarrow{\mathsf{id}\uplus\forall_\vartheta(p)} & \exists_S(q) \uplus \forall_T(p) & \xleftarrow{\exists_\vartheta(q)\uplus\mathsf{id}} & \exists_T(q) \uplus \forall_T(p)
\end{array}$$

$$\begin{array}{ccccc}
\exists_S(p) \otimes \forall_S(q) & \xleftarrow{\exists_\vartheta(q)\otimes\mathsf{id}} & \exists_T(p) \otimes \forall_S(q) & \xrightarrow{\mathsf{id}\otimes\forall_\vartheta(p)} & \exists_T(p) \otimes \forall_T(q) \\
\downarrow & & & & \downarrow \\
\exists_S(p \otimes q) & & \xleftarrow{\qquad\exists_\vartheta(p\otimes q)\qquad} & & \exists_T(p \otimes q) \\
\uparrow & & & & \uparrow \\
\forall_S(p) \otimes \exists_S(q) & \xleftarrow{\mathsf{id}\otimes\exists_\vartheta(q)} & \forall_S(p) \otimes \exists_T(q) & \xrightarrow{\forall_\vartheta(p)\otimes\mathsf{id}} & \forall_T(p) \otimes \exists_T(q)
\end{array}$$

Theorem: a morphism of ldqs induces 2-cells

$$
\begin{array}{ccccc}
s \mathbin{\check{\otimes}} s & \xrightarrow{\omega \check{\otimes} \mathrm{id}} & t \mathbin{\check{\otimes}} s & \xleftarrow{\mathrm{id} \check{\otimes} \psi} & t \mathbin{\check{\otimes}} t \\
\delta \uparrow & & \Uparrow & & \uparrow \delta \\
s & & \xrightarrow{\qquad\qquad \omega \qquad\qquad} & & t \\
\delta \downarrow & & \Downarrow & & \downarrow \delta \\
s \mathbin{\check{\otimes}} s & \xrightarrow{\mathrm{id} \check{\otimes} \omega} & s \mathbin{\check{\otimes}} t & \xleftarrow{\psi \check{\otimes} \mathrm{id}} & t \mathbin{\check{\otimes}} t
\end{array}
$$

$$
\begin{array}{ccccc}
s \mathbin{\hat{\otimes}} s & \xleftarrow{\psi \hat{\otimes} \mathrm{id}} & t \mathbin{\hat{\otimes}} s & \xrightarrow{\mathrm{id} \hat{\otimes} \omega} & t \mathbin{\hat{\otimes}} t \\
\mu \downarrow & & \Uparrow & & \downarrow \mu \\
s & & \xleftarrow{\qquad\qquad \psi \qquad\qquad} & & t \\
\mu \uparrow & & \Downarrow & & \uparrow \mu \\
s \mathbin{\hat{\otimes}} s & \xleftarrow{\mathrm{id} \hat{\otimes} \psi} & s \mathbin{\hat{\otimes}} t & \xrightarrow{\omega \hat{\otimes} \mathrm{id}} & t \mathbin{\hat{\otimes}} t
\end{array}
$$

and in the case of a morphism of Frqs, these are identity 2-cells.

Question: why no converse?

Theorem: Let s and t be Frfs $\mathbf{1} \to (\mathbf{Sup}, \otimes, 2, \oslash, 2^{\mathsf{op}})$, and

$$s \underset{\psi}{\overset{\omega}{\rightleftarrows}} t$$

an arbitrary morphism of lfs, then

$$s \underset{\psi^\sharp}{\overset{\omega}{\rightrightarrows}} t$$

define a lf between s and t (now regarded as ldcs).

We do not get $\omega = \psi^\sharp$ (or even $\omega \leqslant \psi^\sharp$) in general.

[$\omega \leqslant \psi^\sharp \iff \psi 1 \leqslant 1$.]

Definition: a *point* of a lf T is a nat $\forall_T \xrightarrow{\tau} \exists_T$ satisfying

$$\begin{array}{ccccc}
\forall_T(p) \owedge \forall_T(q) & \xrightarrow{\mu} & \forall_T(p \owedge q) & \xleftarrow{\mu} & \forall_T(p) \owedge \forall_T(q) \\
{\scriptstyle \tau_p \owedge \mathsf{id}}\downarrow & & \downarrow{\scriptstyle \tau_{p \owedge q}} & & \downarrow{\scriptstyle \mathsf{id} \owedge \tau_q} \\
\exists_T(p) \owedge \forall_T(q) & \longrightarrow & \exists_T(p \owedge q) & \longleftarrow & \forall_T(p) \owedge \exists_T(q)
\end{array}$$

$$\begin{array}{ccccc}
\forall_T(p) \ovee \exists_T(q) & \longleftarrow & \forall_T(p \ovee q) & \longrightarrow & \exists_T(p) \ovee \forall_T(q) \\
{\scriptstyle \tau_p \ovee \mathsf{id}}\downarrow & & \downarrow{\scriptstyle \tau_{p \ovee q}} & & \downarrow{\scriptstyle \mathsf{id} \ovee \tau_q} \\
\exists_T(p) \ovee \exists_T(q) & \xleftarrow{\delta} & \exists_T(p \ovee q) & \xrightarrow{\delta} & \exists_T(p) \ovee \exists_T(q)
\end{array}$$

A *Frobenius functor* (Frf) is a lf T for which the identity nat is a point. (Structure, not property!)

Lemma: a point is uniquely determined by $e \xrightarrow{\eta} \forall_T(e) \xrightarrow{\tau_e} \exists_T(e)$ (and also by $\forall_T(d) \xrightarrow{\tau_d} \exists_T(d) \xrightarrow{\varepsilon} d$).

Definition: a morphism of pointed lfs $(S, \sigma) \to (T, \tau)$ is a morphism of lfs $\vartheta : S \to T$ which further satisfies

$$\begin{array}{ccc} \forall_S(p) & \xrightarrow{\sigma_p} & \exists_S(p) \\ {\scriptstyle \forall_\vartheta(p)}\big\downarrow & & \big\uparrow{\scriptstyle \exists_\vartheta(p)} \\ \forall_T(p) & \xrightarrow{\tau_p} & \exists_T(p) \end{array}$$

Lemma: it suffices to check

$$\begin{array}{ccccc} e & \xrightarrow{\eta} & \forall_S(e) & \xrightarrow{\sigma_e} & \exists_S(e) \\ & {\scriptstyle \eta}\searrow & & & \big\uparrow{\scriptstyle \exists_\vartheta(e)} \\ & & \forall_T(e) & \xrightarrow{\tau_e} & \exists_T(e) \end{array}$$

(or, equivalently, the d-case).

References:

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