Notes on operator spaces and column Hilbert spaces (INCOMPLETE)

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January 20, 2014

1 Summary

The category of operator spaces and linear complete contractions, **Oper**, resembles that of Banach spaces and linear contractions, **Ban**, except that its structure is yet more intricate! In particular, **Oper** admits three monoidal structures, here denoted $\hat{\otimes}$, $\boxtimes$, and $\check{\otimes}$, all having unit $\mathbb{C}$, and comparison maps $\hat{\otimes} \to \boxtimes \to \check{\otimes}$.

The first and last of these, $\hat{\otimes}$ and $\check{\otimes}$, are analogous to their namesakes on **Ban**. In particular, $\hat{\otimes}$ is closed; $\mathbb{C}$ is a dualising object for $(\mathbf{Oper}_{fd}, \hat{\otimes}, \multimap)$; $\check{\otimes}$ is an extension to the whole of **Oper** of $\hat{\otimes}$'s dual in $\mathbf{Oper}_{fd}$; there are linear distributions $P \hat{\otimes} (Q \check{\otimes} R) \to (P \hat{\otimes} Q) \check{\otimes} R$ relating for arbitrary operator spaces P, Q, R.

But $\boxtimes$ is something quite different and new, and particularly useful with regard to Hilbert spaces. It is non-symmetric in the sense that there exist operator spaces P, Q for which $P \boxtimes Q \ncong Q \boxtimes P$. But it does define a *Frobenius functor*

$$(\mathbf{Oper}, \hat{\otimes}, \mathbb{C}, \check{\otimes}, \mathbb{C}) \times (\mathbf{Oper}, \hat{\otimes}, \mathbb{C}, \check{\otimes}, \mathbb{C}) \to (\mathbf{Oper}, \hat{\otimes}, \mathbb{C}, \check{\otimes}, \mathbb{C})$$

—that is, a monoidal functor $(\mathbf{Oper}, \hat{\otimes}, \mathbb{C}) \times (\mathbf{Oper}, \hat{\otimes}, \mathbb{C}) \to (\mathbf{Oper}, \hat{\otimes}, \mathbb{C})$ together with a compatible comonoidal functor $(\mathbf{Oper}, \check{\otimes}, \mathbb{C}) \times (\mathbf{Oper}, \check{\otimes}, \mathbb{C}) \to (\mathbf{Oper}, \check{\otimes}, \mathbb{C})$.

In particular, there are (non-invertible) comparison maps

$$(P \boxtimes Q) \hat{\otimes} (R \boxtimes S) \to (P \hat{\otimes} R) \boxtimes (Q \hat{\otimes} S) \tag{1}$$

$$(P \check{\otimes} R) \boxtimes (Q \check{\otimes} S) \to (P \boxtimes Q) \check{\otimes} (R \boxtimes S) \tag{2}$$

as well as maps

$$(P \multimap R) \boxtimes (Q \multimap S) \to (P \boxtimes Q) \multimap (R \boxtimes S) \tag{3}$$

induced by (1).

Hilbert spaces can be construed as operator spaces in several inequivalent ways: we shall focus on what are called "column Hilbert spaces".

2 Definitions

A *matrix norm* on a (real or complex) vector space V is a function $\| \ \| : \coprod_{d \in \mathbb{N}} V^{d \times d} \to [0, \infty)$—equivalently, a sequence of functions $\left(\| \ \|_d : V^{d\times d} \to [0, \infty)\right)_{d \in N}$—satisfying

- for all $\zeta \in \mathbb{C}^{d \times e}$, $\alpha \in p^{d \times d}$ and $\omega \in \mathbb{C}^{e \times d}$,

$$\|\zeta \cdot \alpha \cdot \omega\|_e \leq \|\zeta\| \cdot \|\alpha\|_d \cdot \|\omega\|$$

- for every $\alpha \in p^{d\times d}$ and every $\beta \in p^{e\times e}$,

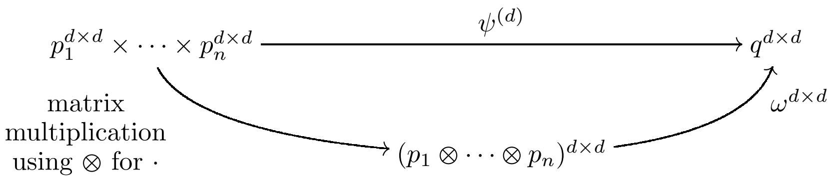
$$\left\| \begin{matrix} \alpha & 0 \\ 0 & \beta \end{matrix} \right\|_{d+e} \leq \max\left\{ \|\alpha\|_d , \|\beta\|_e \right\}$$

in addition to the usual axioms for a norm. (That is, each $\| \ \|_d$ should be a norm on $V^{d\times d}$.) Given a matrix norm, V is Cauchy-complete wrt $\| \ \|_1$ iff $V^{d\times d}$ wrt $\| \ \|_d$, for all d. In this case we say simply that V is Cauchy-complete wrt $\| \ \|$.

An *operator space* P is a complex vector space p together with a matrix norm $\| \ \|_P = (\| \ \|_{P,d})_{d\in\mathbb{N}}$ for which p is Cauchy-complete. We write $\mathcal{M}_d(P)$ for the Banach space comprising $p^{d\times d}$ and $\| \ \|_{P,d}$, and $\widetilde{P}$ for the operator space comprising $\overline{p}$ and the *opposite* matrix norm $\|\alpha\|_{\widetilde{P}} = \left\|\alpha^T\right\|_P$.

A *linear complete contraction* $P \to Q$ is a linear transformation $\omega : p \to q$ such that, for every $d \in \mathbb{N}$, $\omega^{d\times d} : p^{d\times d} \to q^{d\times d}$ defines a linear contraction $\mathcal{M}_d(P) \to \mathcal{M}_d(Q)$. [Here $\omega^{d\times d}$ denotes entry-wise application of ω.] Thus, for each $d \in \mathbb{N}$ there is a forgetful functor $\mathcal{M}_d : \mathbf{Oper} \to \mathbf{Ban}$, where **Oper** denotes the category of operator spaces and linear complete contractions.

There are two distinct notions of "multilinear complete contraction" $P_1 \times \cdots \times P_n \to Q$, based on two different notions of matrix multiplication: the usual one (representing composition of linear transformations), and the Kronecker product (representing the tensor product of linear transformations). The one which appeared first in the literature, and therefore seized control of the name "multilinear complete contraction" is based on composition of matrices; it is that of a multilinear transformation $\psi : p_1 \times \cdots \times p_n \to q$ (with corresponding linear transformation $\omega : p_1 \otimes \cdots \otimes p_n \to q$) such that, for all d, the "(not-so-)obvious map"

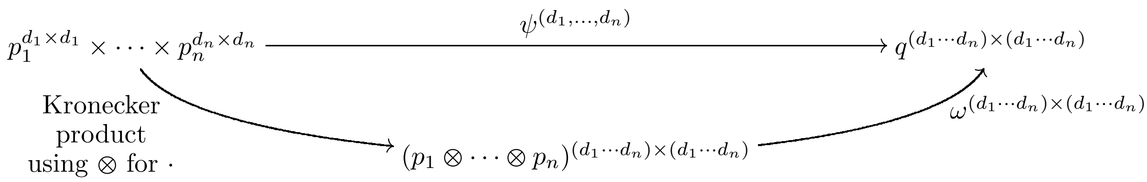


defines a multilinear contraction $\mathcal{M}_d(P_1) \times \cdots \times \mathcal{M}_d(P_n) \to \mathcal{M}_d(Q)$. Explicitly, in the case $n = 2$,

$$[\psi^{(d)}(\alpha, \beta)]_{jk} = \omega\left(\sum_l \alpha_{jl} \otimes \beta_{lk}\right) = \sum_l \psi(\alpha_{jl}, \beta_{lk}).$$

We shall refer to this concept as *multilinear complete-*$\circ$*-contraction* in order to emphasise the role of composition (of matrices) in its definition.

The second concept to appear in the literature happens to be more general, and is saddled with the arguably unfortunate[1] name "multilinear jointly complete contraction"; it is that of a multilinear transformation $\psi : p_1 \times \cdots \times p_n \to q$ (with corresponding linear transformation $\omega : p_1 \otimes \cdots \otimes p_n \to q$) such that, for all $d_1, \ldots, d_n$, the "perhaps-even-less-obvious map"



defines a multilinear contraction $\mathcal{M}_{d_1}(P_1) \times \cdots \times \mathcal{M}_{d_n}(P_n) \to \mathcal{M}_{d_1\cdots d_n}(Q)$. Explicitly, in the case $n = 2$,

$$[\psi^{(d,e)}(\alpha, \beta)]_{jk} = \omega(\alpha_{j'k'} \otimes \beta_{j''k''}) = \psi(\alpha_{j'k'}, \beta_{j''k''})$$

[1] At least to me, "jointly complete" sounds like a more restrictive condition than "complete", whereas—in the case at hand—it proves to be a less restrictive condition.

where $j' = \lfloor \frac{j}{e} \rfloor$, $j'' = j - j'e$, $k' = \lfloor \frac{k}{e} \rfloor$, $k'' = k - k'e$. We shall refer to this concept as *multilinear complete-⊗-contraction* in order to emphasise the role of Kronecker product in its definition. [[This terminology of mine also has some mnemonic side-benefits.]]

Operator spaces and multilinear complete-⊗-contractions form a symmetric multicategory, $\mathbf{Oper}_{\otimes\mathsf{multi}}$, whereas operator spaces and multilinear complete-∘-contractions form a multicategory $\mathbf{Oper}_{\circ\mathsf{multi}}$ which is involutive wrt $\widetilde{(\ )}$, but not symmetric; both happen to be representable.

To explain this, let us consider a bilinear transformation $\psi : p \times q \to r$ (where, as usual, p, q, r underlie operator spaces P, Q, R), and the corresponding bilinear transformation $\chi : q \times p \to r$, defined by $\chi(\beta, \alpha) = \psi(\alpha, \beta)$. Then every entry of $\psi^{(d,e)}(\alpha, \beta)$ appears in $\chi^{(e,d)}(\beta, \alpha)$, and vice versa. More specifically, we have

$$[\psi^{(d,e)}(\alpha, \beta)]_{(j'e+j'')(k'e+k'')} = \psi(\alpha_{j'k'}, \beta_{j''k''}) = \chi(\beta_{j''k''}, \alpha_{j'k'}) = [\chi^{(e,d)}(\beta, \alpha)]_{j''d+j', k''d+k'}$$

—hence, there exists a permutation matrix π such that $\psi^{(d,e)}(\alpha, \beta) = \pi^{-1}\chi^{(e,d)}(\beta, \alpha)\pi$. It follows immediately that $\left\|\psi^{(d,e)}(\alpha, \beta)\right\|_{R,de} = \left\|\chi^{(e,d)}(\beta, \alpha)\right\|_{R,de}$; so ψ is a bilinear complete-⊗-contraction iff χ is so.

On the other hand, $\psi^{(d)}(\alpha, \beta)$ has entries which need not appear in $\chi^{(d)}(\beta, \alpha)$. Rather, one has

$$\begin{aligned} [\psi^{(d)}(\alpha, \beta)]_{jk} &= \sum_l \psi(\alpha_{jl}, \beta_{lk}) \\ &= \sum_l \chi(\beta_{lk}, \alpha_{jl}) \\ &= \sum_l \chi(\beta^T_{kl}, \alpha^T_{lj}) \\ &= [\chi^{(d)}(\beta^T, \alpha^T)]_{kj} \end{aligned}$$

—or, in other words, $\psi^{(d)}(\alpha, \beta) = \left(\chi^{(d)}(\beta^T, \alpha^T)\right)^T$. Hence, ψ defines a bilinear complete-∘-contraction $P \times Q \to R$ iff χ defines a bilinear complete-∘-contraction $\widetilde{Q} \times \widetilde{P} \to \widetilde{R}$. More generally, there is a well-behaved bijective correspondence between multilinear complete-∘-contractions of the form $P_1 \times \cdots \times P_n \to Q$, and those of the form $\widetilde{P_n} \times \cdots \times \widetilde{P_1} \to \widetilde{Q}$, and this is what we mean by an "involutive multicategory".

As stated previously, there is both a universal bilinear complete-⊗-contraction $P \times Q \to P \otimes Q$, and a universal bilinear complete-∘-contraction $P \times Q \to P \boxtimes Q$. These are each defined by taking the Cauchy-completion of $p \otimes q$ wrt the least matrix norm making $\otimes : p \times q \to p \otimes q$ completely(-∘- or -⊗-)contractive. An explicit description of these norms is sufficiently intricate that we postpone it to section **??**; but it should be clear that, if there is any matrix norm making $\otimes : p \times q \to p \otimes q$ completely(-∘- or -⊗-)contractive, then there is also a least one.

Now let us consider the Curry $\phi : q \to p \multimap r$ of (the linear transformation $\omega : p \otimes q \to r$ corresponding to) a bilinear transformation $\psi : p \times q \to r$. Then a direct computation shows that the Curry of $\psi^{(d,1)} : p^{d\times d} \times q^{1\times 1} \to r^{d\times d}$ (defined above) equals the composite

$$q \xrightarrow{\ \ \phi\ \ } p \multimap r \xrightarrow{\ (\ )^{d\times d}\ } p^{d\times d} \multimap r^{d\times d}$$

—namely, for $\alpha = (\alpha_{jk}) \in p^{d\times d}$ and $\beta \in q$, we have

$$\begin{aligned} \psi^{(d,1)}(\alpha, \beta) &= (\psi(\alpha_{jk}, \beta)) \\ &= (\phi(\beta)(\alpha_{jk})) \\ &= \phi(\beta)^{d\times d}(\alpha). \end{aligned}$$

Hence, if $\psi^{(d,1)}$ defines a bilinear contraction $\mathcal{M}_d(P) \times \mathcal{M}_1(Q) \to \mathcal{M}_d(R)$ for all $d \in \mathbb{N}$ (as required by completely-⊗-contractivity), then the range of ϕ lands inside $s = \left\{\omega \in p \multimap r \;\middle|\; \|\omega\|_{S,1} < \infty\right\}$ where

$$\|\omega\|_{S,1} = \sup\left\{\left\|\omega^{d\times d}\right\|_{\mathcal{M}_d(P) \multimap \mathcal{M}_d(R)} \;\middle|\; d \in \mathbb{N}\right\}$$

and, moreover, ϕ is a contraction $\mathcal{M}_1(Q) \to (s, \|\ \|_{S,1})$.

A similar computation shows that the Curry of $\psi^{(1,e)} : p^{1\times 1} \times q^{e\times e} \to r^{e\times e}$ equals the composite

$$q^{e\times e} \xrightarrow{\phi^{e\times e}} (p \multimap r)^{e\times e} \overset{\sim}{\to} p \multimap (r^{e\times e})$$

and, more generally, the Curry of $\psi^{(d,e)} : p^{d\times d} \times q^{e\times e} \to r^{de\times de}$ equals the composite

$$q^{e\times e} \xrightarrow{(\phi^{[d,1]})^{e\times e}} (p^{d\times d} \multimap r^{d\times d})^{e\times e} \overset{\sim}{\to} p^{d\times d} \multimap ((r^{d\times d})^{e\times e}) \overset{\sim}{\to} p^{d\times d} \multimap r^{de\times de}$$

where $\phi^{[d,1]}$ denotes the Curry of $\psi^{(d,1)}$ discussed above. Thus $\|\ \|_{S,1}$ extends to a matrix norm on s,

$$\|\omega\|_{S,e} = \sup\left\{ \|\omega^{d\times d}\|_{\mathcal{M}_d(P)\multimap\mathcal{M}_{de}(R)} \;\middle|\; d \in \mathbb{N} \right\},$$

and ϕ defines a linear complete contraction $Q \to P \multimap R := (s, \|\ \|_S)$ iff ψ is a bilinear complete-$\otimes$-contraction $P \times Q \to R$.

3 Examples

Let $\mathbf{h} = (h, \langle -, - \rangle_{\mathbf{h}})$ be a Hilbert space, and $H = (h, \|\ \|_H)$ its induced Banach space, as in the previous set of notes. For $d \in \mathbb{N}$, let $\mathbf{h}^d$ denote the Hilbert space comprising h^d and the obvious inner product.

$$\left\langle \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_d \end{pmatrix} \right\rangle_{\mathbf{h}^d} = \langle \alpha_1, \beta_1 \rangle_{\mathbf{h}} + \cdots + \langle \alpha_d, \beta_d \rangle_{\mathbf{h}}$$

We denote the induced Banach space of $\mathbf{h}^d$ by $H^{\boxplus d}$, and we define $\mathbf{h} \bullet\!\!- \mathbf{h}$ to be the operator space with $\mathcal{M}_d(\mathbf{h} \bullet\!\!- \mathbf{h}) = H^{\boxplus d} \circ\!\!- H^{\boxplus d}$. We leave as an exercise to the reader that this does define an operator space.

For every Hilbert space $\mathbf{h}$, composition defines a bilinear complete-$\circ$-contraction

$$(\mathbf{h} \bullet\!\!- \mathbf{h}) \times (\mathbf{h} \bullet\!\!- \mathbf{h}) \to \mathbf{h} \bullet\!\!- \mathbf{h}$$

thus also a linear complete contraction

$$(\mathbf{h} \bullet\!\!- \mathbf{h}) \boxtimes (\mathbf{h} \bullet\!\!- \mathbf{h}) \to \mathbf{h} \bullet\!\!- \mathbf{h}.$$

Similarly, adjunction defines a linear complete contraction $\dagger : \widetilde{\mathbf{h} \bullet\!\!- \mathbf{h}} \to \mathbf{h} \bullet\!\!- \mathbf{h}$.

Every operator space can be embedded into one of the form $\mathbf{h} \bullet\!\!- \mathbf{h}$, and every monoid in $(\mathbf{Oper}, \boxtimes, \mathbb{C})$ can be embedded into one of the form $(\mathbf{h} \bullet\!\!- \mathbf{h}, \circ, \mathrm{id})$. But not every dagger monoid in $(\mathbf{Oper}, \boxtimes, \widetilde{(\ )}, \mathbb{C})$ can be embedded into one of the form $(\mathbf{h} \bullet\!\!- \mathbf{h}, \circ, \dagger, \mathrm{id})$.