Non-Hausdorff topologies and polar co-ordinates: a divertissement

(notes for a talk by JME)

March 4, 2016

Background

Email to Bob: three talks in three weeks, all on different topics? We agreed that I should go with this 'light' talk first (March 1); another, more serious, topic will follow next week (March 8); possibly also the third topic on March 15. (But perhaps the second topic will require more than one week?)

This talk: toying around with the definition of (the locale) $\mathbb{R}$ in an elementary topos, led to a very minor epiphany about polar co-ordinates for complex numbers: they describe $(\mathbb{C}, \cdot, 1)$ as a pullback in the category of topological monoids.

But polar co-ordinates are not just for complex numbers—also complex matrices, bounded operators on a Hilbert space! I think that they probably also describe $\mathbb{C}^{m\times n}$ as a pullback in the category of topological spaces, but I haven't proven it yet—I attach no priority to doing so, though some of the ideas which emerge are fun and may prove useful in another context.

But we begin with numbers.

Polar decomposition for complex numbers

Undergrads are generally taught (at best) something like:

> *Every non-zero complex number z can be factored uniquely as $r \cdot e^{i\theta}$ where r is a positive real number and θ an element of $\mathbb{R}/2\pi\mathbb{R}$.*

—in other words, the punctured plane $\mathbb{C} \setminus \{0\}$ is isomorphic (in several categories—most obviously, that of topological groups) to the cartesian product of $(0, \infty)$ and $\mathbb{R}/2\pi\mathbb{R}$.

But how do we account for zero? It's usually brushed under the carpet.

Handling zero without topology (old stuff)

As mere sets, we can say that $\mathbb{C}$ is isomorphic to $1 + (0, \infty) \times \mathbb{R}/2\pi\mathbb{R}$; but that's not really impressive since, for instance, $\mathbb{C}$ is also isomorphic to $\mathbb{N}^{\mathbb{N}}$ in the category of sets. But any set of the form $1 + A \times B$ is correctly viewed as a pullback!

$$\begin{array}{ccc} 1 + A \times B & \longrightarrow & 1 + A \\ \downarrow & & \downarrow \\ 1 + B & \longrightarrow & 1 + 1 \end{array}$$

(In general, + commutes with pullbacks in **Set**.)

So the right way of dealing with 0 is to add, not merely a "length zero", but also an "undefined angle" $\perp$, so that 0 can be written as $0 \cdot e^{i\perp}$. The grammar of pullbacks prevents the occurrence of expressions such as $0 \cdot e^{i\theta}$ (where $\theta \in \mathbb{R}/2\pi\mathbb{R}$) and $r \cdot e^{i\perp}$ (where $r > 0$).

To ensure that we still have a pullback of monoids (not merely sets) one must define $\perp + \theta = \perp$. In light of this, we should define $e^{i\perp} = 0$ so that we still have a homomorphism $\{\perp\} \cup \mathbb{R}/2\pi\mathbb{R} \to \mathbb{C}$. In fact, we might as well identify $\{\bot\} \cup \mathbb{R}/2\pi\mathbb{R}$ with the range of said homomorphism, $\{0\} \cup S^1$. Note that the intersection of $\{0\} \cup S^1$ with $[0, \infty)$ is just $\{0, 1\}$.

So we actually have two pullbacks simultaneously.

$$\begin{array}{ccc} \mathbb{C} & \overset{|\,|}{\twoheadrightarrow} & [0,\infty) \\ & \leftarrowtail & \\ \text{arg}\downarrow\!\!\!\!\twoheadrightarrow \ \uparrow & & \uparrow\ \downarrow\text{arg} \\ & \leftarrowtail & \\ \{0\}\cup S^1 & \underset{|\,|}{\twoheadrightarrow} & \{0,1\} \end{array}$$

[In fact, now that I come to think about it, there are actually four pullback squares in that diagram. . .]

Handling zero with topology (new stuff)

If we topologise $\{0, 1\}$ and $\{0\} \cup S^1$ correctly, then

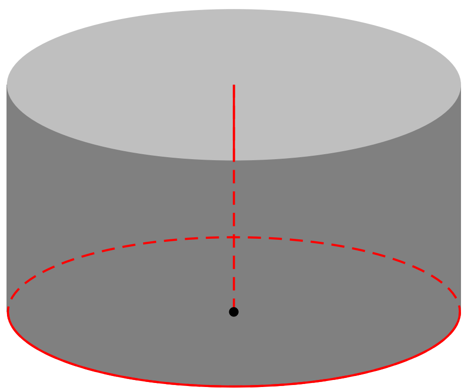
$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{|\,|} & [0,\infty) \\ \text{arg}\big\downarrow & & \big\downarrow\text{arg} \\ \{0\}\cup S^1 & \xrightarrow[|\,|]{} & \{0,1\} \end{array}$$

remains a pullback of topological monoids.

We topologise $[0, \infty)$ in the usual way; so, in order for $\text{arg}\colon [0, \infty) \to \{0, 1\}$ to be continuous, $\{0, 1\}$ needs to be given the Sierpinski topology, with 0 is closed and 1 is open. Similarly, we topologise $\{0\} \cup S^1$ so that the only open neighbourhood of 0 is the whole space; then $\text{arg}\colon \mathbb{C} \to \{0\} \cup S^1$ is continuous.

In both cases, multiplication is continuous wrt the given topologies; moreover, the topology induced on $\mathbb{C}$ as a subspace of $[0, \infty) \times (\{0\} \cup S^1)$ is the usual one; so we have a pullback in the category of topological monoids.

Here's a picture:



the product $[0, \infty) \times (\{0\} \cup S^1)$ is a cylinder plus a line; the pullback $[0, \infty) \times_{\{0,1\}} (\{0\} \cup S^1)$ omits the red circle and the red part of the line, leaving just the black point (the origin) and the rest of the cylinder.

Note that, while

$$\begin{array}{ccc} \mathbb{C} & \leftarrowtail & [0,\infty) \\ \text{arg}\big\downarrow & & \big\downarrow\text{arg} \\ \{0\}\cup S^1 & \leftarrowtail & \{0,1\} \end{array}$$

is also still a pullback in topological monoids, the inclusions $\{0,1\} \to [0,\infty)$ and $\{0\} \cup S^1 \to \mathbb{C}$ are no longer continuous.

Polar decomposition of complex matrices

(Equivalently, polar decomposition of linear transformations between finite-dimensional Hilbert spaces.)

Most of the results in this section generalise to linear transformations between arbitrary Hilbert spaces, but in a few cases the proofs become more difficult, and the language becomes less familiar to a general audience; hence we focus on the finite-dimensional case.

Easy case

> *Every invertible $n \times n$ matrix A can be factored uniquely as $U \circ P$, where U is unitary and P is positive.*

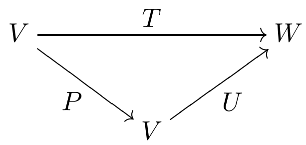
[Unitaries are those matrices satisfying $A^\dagger = A^{-1}$ (obvious generalisation of unit complex numbers); equivalently, they are normal matrices ($A^\dagger \circ A = A \circ A^\dagger$) whose eigenvalues are all unit complex numbers. ($A^\dagger$ denotes the Hermitian transpose $\overline{A}^T$.) Positives are those matrices of the form $B^\dagger \circ B$, for some B (obvious generalisation of non-negative reals); equivalently, they are self-adjoint matrices ($A^\dagger = A$) whose eigenvalues are all non-negative. (Self-adjoint matrices are the same as normal matrices with real eigenvalues.) I use $\circ$ to denote matrix multiplication to remind myself that we are (implicitly) writing composition in the Polish—*i.e.*, non-diagrammatic—order.]

We take the "prove uniqueness first" approach: if $A = U \circ P$, then

$$\begin{aligned} A^\dagger \circ A &= (U \circ P)^\dagger \circ (U \circ P) \\ &= P \circ U^\dagger \circ U \circ P \\ &= P \circ P \end{aligned}$$

—now $A^\dagger \circ A$, being both positive and invertible, has precisely 2^n square roots; but only one of these is again positive.

Note that if we rephrase this proof in terms of an invertible linear transformation between arbitrary finite-dimensional Hilbert spaces, $T\colon V \to W$, it still works; but P is—both by construction and by definition—an endomorphism of V.



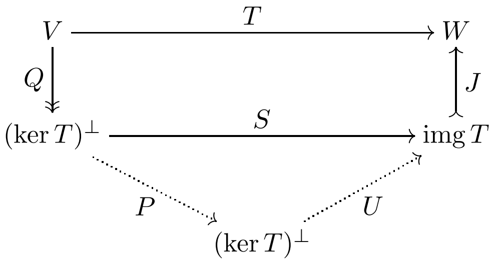
So this factorisation is unique "on the nose", not merely unique up to isomorphism.

General case

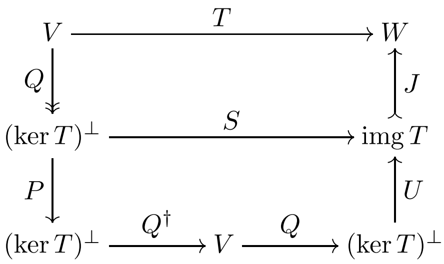
It is actually more convenient to treat the general case of complex $m \times n$ matrices by using the language of linear transformations between arbitrary finite-dimensional Hilbert spaces, $T\colon V \to W$, as in the previous paragraph. (We do not assume V and W to have the same dimension.)

Let J denote the inclusion of $\operatorname{img} T$ in W, and Q the quotient map $V \to V/\ker T \cong (\ker T)^\perp$. (Note that $J^\dagger \circ J$ and $Q \circ Q^\dagger$ are identities.)

We invoke Noether's first isomorphism theorem to obtain an invertible linear transformation $S\colon (\ker T)^{\perp} \cong V/\ker T \to \operatorname{img} T$, and apply the previous factorisation result to it.



Since we're looking for an endomorphism of V, we add a superfluous $Q \circ Q^{\dagger}$, as follows.



Now $\tilde{P} := Q^{\dagger} \circ P \circ Q$ is still positive;

$$P = B^{\dagger} \circ B \;\Rightarrow\; Q^{\dagger} \circ P \circ Q = (B \circ Q)^{\dagger} \circ (B \circ Q)$$

indeed, it is still the unique positive square root of $T^{\dagger} \circ T$.

$$\begin{aligned} T^{\dagger} \circ T &= (J \circ S \circ Q)^{\dagger} \circ (J \circ S \circ Q) \\ &= Q^{\dagger} \circ (S^{\dagger} \circ (J^{\dagger} \circ J) \circ S) \circ Q \\ &= Q^{\dagger} \circ (S^{\dagger} \circ S) \circ Q \\ &= Q^{\dagger} \circ (P \circ P) \circ Q \\ &= (Q^{\dagger} \circ P \circ Q) \circ (Q^{\dagger} \circ P \circ Q) \end{aligned}$$

Note also that, since P and $Q^{\dagger}$ are both injective, $\ker(Q^{\dagger} \circ P \circ Q) = \ker Q = \ker T$.

But what of $\tilde{U} := J \circ U \circ Q\colon V \to W$? Writing $V = (\ker T)^{\perp} \oplus (\ker T)$ and $W = (\operatorname{img} T) \oplus (\operatorname{img} T)^{\perp}$, we get

$$\tilde{U} = \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}$$

—such maps are, naturally enough, called *partial isometries*; they are characterised by the formula $R \circ R^{\dagger} \circ R = R$. It is clear that a complex number satisfies the formula $\omega \cdot \overline{\omega} \cdot \omega = \omega$ iff it belongs to $\{0\} \cup S^1$; so partial isometries are a reasonable generalisation of what we want.

For an arbitrary partial isometry R, both $R^{\dagger} \circ R$ and $R \circ R^{\dagger}$ are *projections*—*i.e.*, self-adjoint idempotents. There is a well-known bijective correspondence between projections $V \to V$ and (linear) subspaces of V; in the case of a partial isometry R, $R^{\dagger} \circ R$ and $R \circ R^{\dagger}$ correspond to the "domain" and "range" of isometry at its core.

(For example, in the case of $\tilde{U} = J \circ U \circ Q$,

$$\begin{aligned} \tilde{U}^{\dagger} \circ \tilde{U} &= Q^{\dagger} \circ U^{\dagger} \circ J^{\dagger} \circ J \circ U \circ Q \\ &= Q^{\dagger} \circ U^{\dagger} \circ U \circ Q \\ &= Q^{\dagger} \circ Q \end{aligned}$$

corresponds to $(\ker T)^{\perp}$, and

$$\begin{aligned}\tilde{U} \circ \tilde{U}^{\dagger} &= J \circ U \circ Q \circ Q^{\dagger} \circ U^{\dagger} \circ J^{\dagger} \\ &= J \circ U \circ U^{\dagger} \circ J^{\dagger} \\ &= J \circ J^{\dagger}\end{aligned}$$

corresponds to $\operatorname{img} T$.)

We are now ready to state the polar decomposition theorem for arbitrary matrices.

> *Every linear transformation between finite-dimensional Hilbert spaces $T\colon V \to W$, can be decomposed as $\tilde{U} \circ \tilde{P}$ with $\tilde{U}$ a partial isometry and P positive;*
>
> $$\begin{array}{ccc} V & \xrightarrow{\quad T \quad} & W \\ & {\scriptstyle \tilde{P}} \searrow \quad \nearrow {\scriptstyle \tilde{U}} & \\ & V & \end{array}$$
>
> *moreover, if we insist that $\tilde{U}^{\dagger} \circ U$ be the projection corresponding to $(\ker \tilde{P})^{\perp}$, then this factorisation is unique.*

I am not going to prove the second half of the statement, which would require an explicit description of the correspondence between projections and subspaces; indeed I have already wasted much more time on these notes than I intended to.

I just note that this is clearly the statement that the *set* of linear transformations $V \to W$ is a pullback.

$$\begin{array}{ccc} \{\text{arbitrary } V \to W\} & \xrightarrow{\qquad \tilde{P} \qquad} & \{\text{positive } V \to V\} \\ {\scriptstyle \tilde{U}}\big\downarrow & & \big\downarrow{\scriptstyle (\ker -)^{\perp}} \\ \{\text{partial isometries } V \to W\} & \xrightarrow{\text{dom}} & \{\text{projections } V \to V = \text{subspaces of } V\} \end{array}$$

[$\tilde{P}(T) = \sqrt{T^{\dagger} \circ T}$ as discussed above; similarly, $\operatorname{dom}(R) = R^{\dagger} \circ R = \sqrt{R^{\dagger} \circ R}$. So

$$\begin{array}{ccc} \{\text{arbitrary } V \to W\} & \xrightarrow{\qquad \tilde{P} \qquad} & \{\text{positive } V \to V\} \\ \big\uparrow & & \big\uparrow \\ \{\text{partial isometries } V \to W\} & \xrightarrow{\text{dom}} & \{\text{projections } V \to V = \text{subspaces of } V\} \end{array}$$

is also a pullback of sets.]

In the case where $V = W$, we have also that

$$\begin{array}{ccc} \{\text{arbitrary } V \to W\} & \longleftarrow\!\!\!\!\prec & \{\text{positive } V \to V\} \\ \big\uparrow & & \big\uparrow \\ \{\text{partial isometries } V \to W\} & \longleftarrow\!\!\!\!\prec & \{\text{projections } V \to V = \text{subspaces of } V\} \end{array}$$

is a pullback of sets.

But none of these statements can be rephrased in the category of monoids; neither the positives nor the partial isometries are closed under multiplication, so they are not obviously monoids at all.

Speculation

As noted previously, the statement that any two sets are isomorphic in **Set** is pretty unimpressive; but perhaps we can get a pullback square in the category of topological spaces as before? It may not be immediately clear that $\sqrt{T^\dagger \circ T}$ is continuous wrt the usual topologies, but apparently it is.

However, to make

$$\{\text{positives}\} \to \{\text{projections=subspaces}\}$$

continuous we will again need a non-Hausdorff topology: the fibre over H is the space of invertible positives, which is open and not closed. I propose to try to reach the necessary topology via a non-symmetric metric.

Given an *effort space* (=generalised metric space in the sense of Lawvere), (E, ϕ), it is natural to extend ϕ to subsets of E as follows.

$$\Phi(A, B) = \sup_{a \in A} \inf_{b \in B} \phi(a, b)$$

(The usual *Hausdorff metric* on subsets of a metric space is a symmetrised version of this.) [Note that if $A \subseteq B$, then $\Phi(A, B) = 0$, tautologously; on the other hand $\Phi(A, \emptyset) = \infty$, unless $A = \emptyset$. (In fact, $\Phi(A, B) = 0$ iff A is contained in the closure of B relative to the welling topology on E.)]

If E is a Hilbert space (or, more generally, a Banach space), ϕ is the norm-derived effort on E, and A and B are subspaces of E, then

$$\Phi(A, B) = \begin{cases} 0 & \text{if } A \subseteq B \\ \infty & \text{otherwise} \end{cases}$$

—so Φ just encodes the usual lattice structure of the subspaces of E. But it is generally understood that it the unit ball of a Banach space is its true "underlying space", and if we restrict our attention to the unit ball of E, $E_{\leq 1}$,

$$\tilde{\Phi}(A, B) = \Phi(E_{\leq 1} \cap A, E_{\leq 1} \cap B)$$

then a radically different picture emerges.

For every $a \in E_{\leq 1} \cap A$,

$$\inf_{b \in E_{\leq 1} \cap B} \phi(a, b) \leq \phi(a, 0) = \|a\| \leq 1$$

—hence, for all subspaces A and B, we get $\tilde{\Phi}(A, B) \leq 1$. In the case of a Hilbert space, it is easy to see that $\tilde{\Phi}(A, B) = 1$ iff $B^\perp \cap A \neq \{0\}$. Moreover, as previously discussed, $\tilde{\Phi}(A, B) = 0$ iff A is contained in the closure of B—but since B is already closed, that just means $A \subseteq B$. So, in general, we may have $0 < \tilde{\Phi}(A, B) < 1$.

A key feature of the welling topology associated to $\tilde{\Phi}$ is that every open well

$$W_{A,\varepsilon} = \left\{ B \subseteq H \;\middle|\; \tilde{\Phi}(A, B) < \varepsilon \right\}$$

contains $\{B \subseteq H \mid A \subseteq B\}$. In particular, H belongs to every open set except the empty one. At the other extreme, $\{0\} \in W_{A,\varepsilon}$ iff either $A = \{0\}$—in which case, $W_{A,\varepsilon}$ is the whole space—or $\varepsilon > 1$—in which case, $W_{A,\varepsilon}$ is again the whole space. In other words, $\{0\}$ belongs to none of open sets except the whole one.

My intuition is that one can probably do something similar for partial isometries, but I haven't bothered to try working it out.