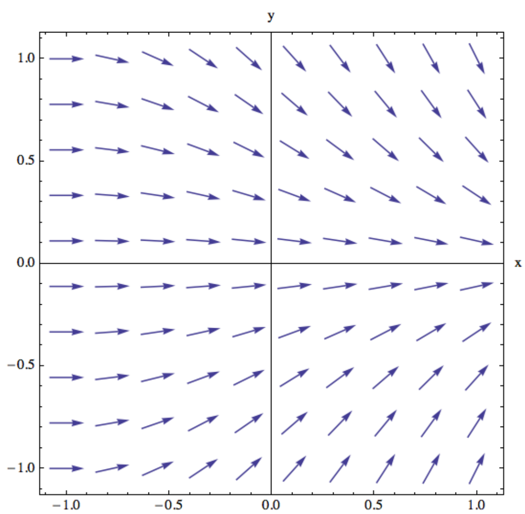
Arts & Science 1D06 Quiz #8 February 12th, 2013

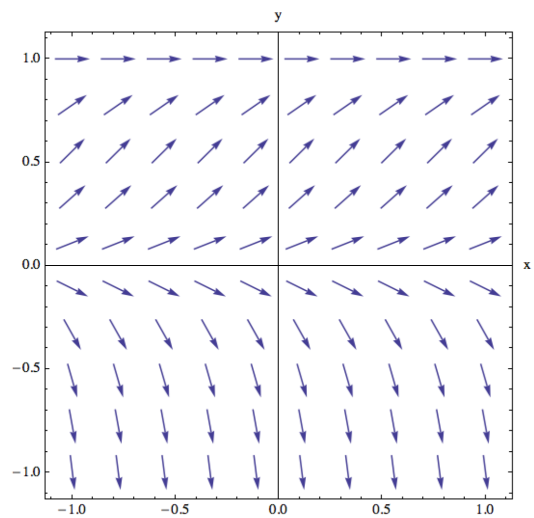
Solutions

1) [4 marks] Match each differential equation with its direction field:

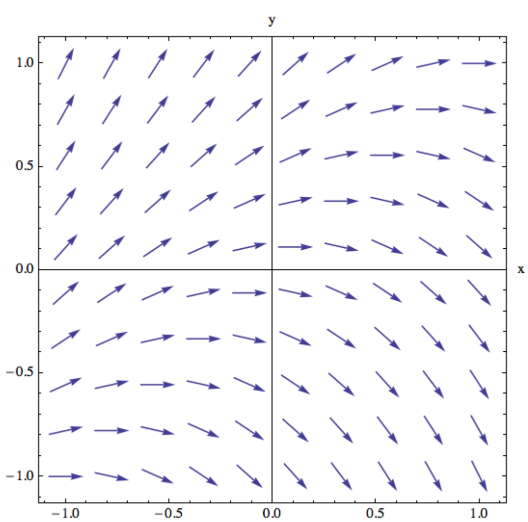
1. $\dfrac{dy}{dx} = y - x$ 2. $\dfrac{dy}{dx} = 4y(1-y)$ 3. $\dfrac{dy}{dx} = e^{y^2/4}\sin(y)$ 4. $\dfrac{dy}{dx} = -y(1+x)$



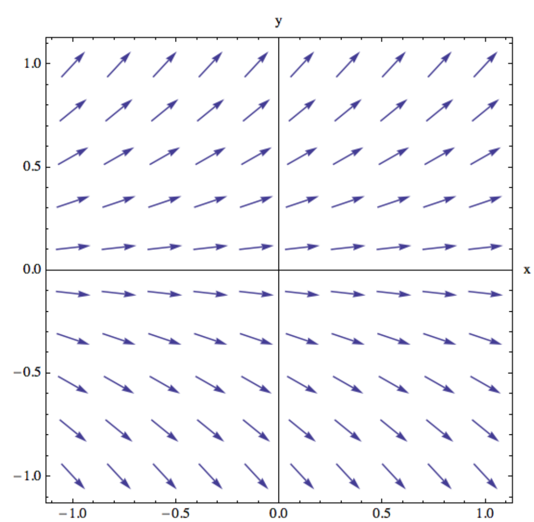
A. 4



B. 2



C. 1



D. 3

2) a) [2 marks] State the general formula for Euler's Method in terms of x_n and y_n with step size h applied to the following initial value problem:

$$y' = F(x, y), \quad y(x_0) = y_0$$

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + hF(x_n, y_n)$$

b) [1 mark] Compute y_1 for the initial value problem $y' = xy - x^2 + y$, $y(0) = 2$ using Euler's Method with step size $h = 1$.

$$\begin{aligned} x_1 &= x_0 + h = 0 + 1 = 1 \\ y_1 &= y_0 + hF(x_0, y_0) \\ &= y_0 + h(x_0 y_0 - x_0^2 + y_0) \\ &= 2 + 1((0)(2) - (0)^2 + (2)) \\ &= 4 \end{aligned}$$

3) [4 marks] Solve the following initial value problem:

$$y' = e^{x+2y} \quad y(0) = 0$$

Notice that

$$y' = \frac{dy}{dx} = e^{x+2y} = e^x e^{2y}.$$

This equation is seperable.

$$\begin{aligned} \frac{dy}{dx} &= e^x e^{2y} \\ \int \frac{1}{e^{2y}} dy &= \int e^x dx \\ \int e^{-2y} dy &= \int e^x dx \\ \frac{e^{-2y}}{-2} &= e^x + c_1 \end{aligned} \tag{1}$$

Now substitute in the initial value, $y(0) = 0$, to solve for the constant of integration, c_1.

$$\begin{aligned} \frac{e^{-2(0)}}{-2} &= e^{(0)} + c_1 \\ -\frac{1}{2} &= 1 + c_1 \\ c_1 &= -\frac{3}{2} \end{aligned}$$

Finally, substitute your now known value of c_1 into the general solution (1) to solve for y explicitly:

$$\begin{aligned}
\frac{e^{-2y}}{-2} &= e^x + \left(-\frac{3}{2}\right) \\
e^{-2y} &= -2\left(e^x + \left(-\frac{3}{2}\right)\right) \\
e^{-2y} &= -2e^x + 3 \\
-2y &= \ln(-2e^x + 3) \\
y &= -\frac{1}{2}(\ln(-2e^x + 3))
\end{aligned}$$

Notice that this solution only holds for $-2e^x + 3 > 0$ so that we can actually take the ln of both sides above to isolate for y.

4) Determine whether the following improper integrals are convergent or divergent.
If the integral is convergent, give the value it converges to.

a) [3 marks] $\displaystyle\int_{-\infty}^{1} \frac{x}{e^{x^2}} dx$

We use substitution to solve this integral:

$$\begin{aligned}
\int_{-\infty}^{1} \frac{x}{e^{x^2}} dx &= \lim_{L\to-\infty} \int_{L}^{1} \frac{x}{e^{x^2}} dx \qquad \text{Let } u = x^2. \text{ Then } du = 2xdx \\
&= \lim_{L\to-\infty} \int_{u=L^2}^{u=1^2} \frac{1}{2e^u} du \\
&= \frac{1}{2} \lim_{L\to-\infty} \int_{L^2}^{1} e^{-u} du \\
&= \frac{1}{2} \lim_{L\to-\infty} -e^{-u}\Big|_{u=L^2}^{u=1} \\
&= -\frac{1}{2}\left(e^1 - \lim_{L\to-\infty} e^{-L^2}\right) \quad (\lim_{L\to-\infty} e^{-L^2} \to 0) \\
&= -\frac{1}{2}e
\end{aligned}$$

b) [3 marks] $\int_0^\infty xe^{-3x}dx$

We use integration by parts to solve this integral:

$$\int_0^\infty xe^{-3x}dx = \lim_{L\to\infty}\int_0^L xe^{-3x}dx$$

Let $u = x$. Then $du = dx$.

Let $dv = e^{-3x}$. Then $v = -\frac{e^{-3x}}{3}$

$$= \lim_{L\to\infty}\left(-\frac{xe^{-3x}}{3}\Big|_0^L\right) - \lim_{L\to\infty}\int_0^L -\frac{e^{-3x}}{3}dx$$

$$= \underbrace{\lim_{L\to\infty}\left(-\frac{Le^{-3L}}{3}\right)}_{\infty\cdot 0 \text{ (indeterminate)}} - \left(-\frac{(0)e^{-3(0)}}{3}\right) + \frac{1}{3}\lim_{L\to\infty}\int_0^L e^{-3x}dx$$

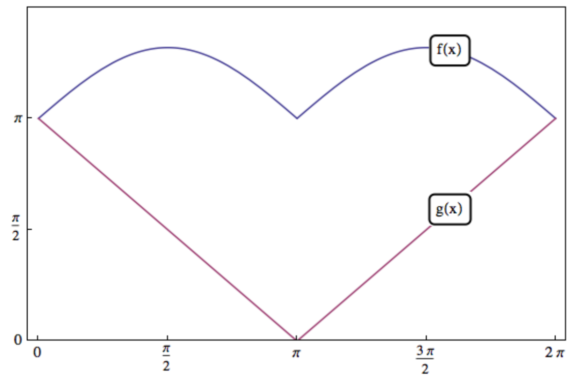
$$= -\frac{1}{3}\underbrace{\lim_{L\to\infty}\left(\frac{L}{e^{3L}}\right)}_{\infty/\infty \text{ (use L'Hopital's Rule)}} + \frac{1}{3}\lim_{L\to\infty}\left(-\frac{e^{-3x}}{3}\Big|_0^L\right)$$

$$= -\frac{1}{3}\lim_{L\to\infty}\frac{1}{3e^{3L}} - \frac{1}{9}\left(\lim_{L\to\infty}e^{-3L} - e^{-3(0)}\right) \quad (\to 0)$$

$$= -\frac{1}{9}\lim_{L\to\infty}e^{-3L} + \frac{1}{9} \quad (\to 0)$$

$$= \frac{1}{9}$$

5) [3 marks] Set up, but **do not evaluate**, an expression to compute the area between these two curves.



$$f(x) = \begin{cases} \sin(x) + \pi, & \text{if } 0 \le x \le \pi; \\ \sin(x-\pi) + \pi, & \text{if } \pi < x \le 2\pi. \end{cases}$$

$$g(x) = |x - \pi| = \begin{cases} -(x-\pi) & \text{if } 0 \le x \le \pi; \\ x - \pi & \text{if } \pi < x \le 2\pi. \end{cases} \quad **$$

$$\text{Area} = \int_0^{\pi} (\sin(x) + \pi - (-(x-\pi)))dx + \int_{\pi}^{2\pi} (\sin(x-\pi) + \pi - (x-\pi))dx$$

**Please note that this piecewise definition was corrected from the quiz where the function pieces were switched (and thus corresponded to the wrong interval).

BONUS Explain why the following differential equation is a poor model for the number of influenza cases $I(t)$ in a population over time:

$$\frac{dI}{dt} = kI \quad \text{where k>0 is some constant}$$

There are many reasons why this differential equation is a poor model for the number of influenza cases in a population over time. For one, notice that this equation corresponds to exponential growth, *i.e.*, the solution to this equation is the exponential function:

$$I(t) = Ce^{kt}$$

where $C > 0$ is some constant. This exponential function grows without bound, that is,

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} Ce^{kt} = \infty.$$

In other words, the model predicts that as time elapses, an infinite number of influenza cases will occur. This prediction is unreasonable since, presumably, we have a finite population, and so this model is a poor one.

But suppose, for the sake of argument, that our population is large enough that it is essentially infinte (very, very, very large). The above is still a poor model for influenza since it predicts that eventually the whole population will be infected and that's the end of the story. However, we know that most people do recover from influenza, which is definitely something important that we should seek to capture with our model but that we haven't accounted for above. Moreover, we know that influenza is seasonal and we seem to have a regular outbreak every year, which is, again, different from what our model predicts. How could we possibly incorporate these known properties of influenza into our model...?