Arts & Science 1D6 Test #2

Day Class
Test #2
Duration of test: 60 minutes
McMaster University
1 March, 2011

Dr. Matt Valeriote

Last Name:______________________________ Initials:__________

Student No.:______________________________ Your TA's Name:______________

This test has 8 pages and 7 questions and is printed on BOTH sides of the paper. Pages 7 and 8 contain no questions and can be used for rough work.

You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancies to the attention of the invigilator.

Attempt all questions and write your answers in the space provided.

Marks are indicated next to each question; the total number of marks is 40.

Any Casio fx991 calculator is allowed. Other aids are not permitted.

Use pen to write your test. If you use a pencil, your test will not be accepted for regrading (if needed).

Good Luck!

Score

Question	1	2	3	4	5	6	7	Total
Points	6	5	6	6	6	6	5	40
Score								

continued ...

ALL QUESTIONS: you must show your work to receive full credit.

1. [6] Let $g(x) = \int_0^{x^2} te^{-t}\, dt$.

   (a) Compute $g'(x)$.

   (b) Find the interval(s) on which the function $g(x)$ is concave upward

2. [5] Determine if the following improper integral is convergent and evaluate it if it is.

$$\int_0^2 \frac{x}{4 - x^2}\, dx.$$

continued . . .

3. [6] Evaluate the following indefinite integrals.

   (a) $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}\,dx$

   (b) $\displaystyle\int 4x\cos(4x)\,dx$

4. [6] Let R be the region bounded by the curve $y = x(x-1)^2$ and the x-axis.

   (a) Find the area of R.

continued ...

(b) Set up but DO NOT EVALUATE the integral representing the volume obtained by rotating the region R about the line $y = -1$.

5. [6] Solve the initial value problem $(x+1)\dfrac{dy}{dx} = y^2$, $y(0) = \frac{1}{2}$ for $x > -1$.

continued ...

6. [6] A cup of coffee is poured from a pot whose contents are 95°C into a non-insulated cup in a room at 20°C. Let $T(t)$ be the temperature of the coffee after t minutes. Assuming that the coffee cools according to Newton's Law, then

$$\frac{dT}{dt} = k(T - 20).$$

(a) Solve this differential equation subject to the initial condition $T(0) = 95$.

(b) After one minute, the coffee has cooled to 90°C. Use this information and your solution to (a) to solve for the constant k.

continued . . .

7. [6] Populations of aphids (A) and ladybugs (L) are modeled by the predator-prey equations

$$\begin{aligned}\frac{dA}{dt} &= 2A - 0.01AL \\ \frac{dL}{dt} &= -0.5L + 0.0001AL\end{aligned}$$

(a) Find the equilibrium solutions and explain their significance.

(b) When there are 1000 aphids and 300 ladybugs, is the aphid population increasing or decreasing? Justify your answer.

continued . . .

Name ______________________________

Student Number ______________________________

Your TA's Name: ______________________________

Arts & Science 1D06

DAY CLASS
APRIL EXAM
DURATION OF EXAM: 3 Hours
MCMASTER UNIVERSITY

DR. MATT VALERIOTE

12 April, 2011

THIS EXAMINATION PAPER INCLUDES 14 PAGES AND 17 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF YOUR INVIGILATOR.
Attempt all questions.
The total number of available points is 100.
Marks are indicated next to each question.
Use of a Casio fx991 calculator only is allowed.
Write your answers in the space provided.
You must show your work to get full credit.
Use the last two pages for rough work.
Good Luck.

Score

Question	1–3	4–6	7	8	9	10	11
Points	9	9	8	6	9	7	7
Score							

Question	12	13	14	15	16	17	Total
Points	8	6	4	10	9	8	100
Score							

Continued on Page 2 . . .

Multiple Choice Questions

Indicate your answers to questions 1–3 by circling only ONE of the letters. Each of these questions is worth 3 marks.

1. [3] $\lim_{x\to 0^+} x^{x^2}$ is equal to

   (A) 0 (B) 1 (C) e (D) does not exist

2. [3] Which of the following three tests can be used to show that the series $\sum_{n=1}^{\infty} \frac{3}{n(n+2)}$ is convergent?

   (I) The Ratio Test.

   (II) The Comparison Test with $\sum_{n=1}^{\infty} 3n^{-2}$.

   (III) The Limit Comparison Test with $\sum_{n=1}^{\infty} 3n^{-1}$.

   (A) none (B) I only (C) II only (D) III only

   (E) I and II (F) I and III (G) II and III (H) all three

Continued on Page 3. . .

3. [3] Let $f(x) = \int_0^{\cos x} \sqrt[3]{1 - t^2}\, dt$. Then $f'(x)$ is:

(A) $\dfrac{1}{3(1 - \cos^2 x)^{2/3}}$ (B) $-\sin x \sqrt[3]{1 - \cos^2 x}$

(C) $\sqrt[3]{1 - \cos^2 x}$ (D) $\dfrac{-2 \sin x \cos x}{3(1 - \cos^2 x)^{2/3}}$

Continued on Page 4...

True/False Questions.

Decide whether the statements in questions 4–6 are true or false by circling your choice. YOU MUST JUSTIFY YOUR ANSWER TO RECEIVE FULL CREDIT. Each of these questions is worth 3 marks.

4. [3] If $f'(x)$ exists and is nonzero for all x then $f(1) \neq f(0)$.

TRUE FALSE

5. [3] The differential equation $y' = x^2 + y^2 + 1$ has an equilibrium solution.

TRUE FALSE

6. [3] If b_n is a sequence of positive numbers such that $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

TRUE FALSE

Continued on Page 5 . . .

Questions 7–17: you must show work to receive full credit

7. Consider the function $f(x) = x\sqrt{1-x^2}$.

(a) [2] Find the x-intercepts of $f(x)$.

(b) [3] Compute $f'(x)$.

(c) [3] On which interval(s) is $f(x)$ increasing?

8. [6] Find the area of the region enclosed by the curves $y = 4 - x^2$ and $y = x^2 + 2$.

Continued on Page 6...

9. Compute the following integrals.

   (a) [2] $\displaystyle\int_3^4 \frac{x}{\sqrt{25-x^2}}\,dx.$

   (b) [3] $\displaystyle\int \ln(1+x^2)\,dx.$ Hint: Use Integration by Parts.

   (c) [4] $\displaystyle\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx.$

Continued on Page 7...

10. The growth of a population of mice is given by the differential equation

$$\frac{dP}{dt} = (0.5)P\left(1 - \frac{P}{500}\right),$$

with time measured in months. Assume that the initial size of the population is 5.

(a) [3] Write a formula for the population $P(t)$. You may use that the general solution of the logistic equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$ is $\dfrac{K}{1 + Ae^{-kt}}$ for some constant A.

(b) [4] How many months does it take for the population to climb to 100?

11.[7] Find the solution of the differential equation $xy\dfrac{dy}{dx} = x + 1$, for $x > 0$, that satisfies the initial condition $y(1) = 2$.

Continued on Page 8...

12. Consider the differential equation

$$xy' = 4x^3 - y.$$

(a) [2] By rewriting this differential equation, show that it is linear.

(b) [3] Find an integrating factor for this differential equation.

(c) [3] Find the solution of this differential equation subject to the condition $y(1) = 0$.

Continued on Page 9. . .

13. Determine whether the sequence converges or diverges. If it converges, find the limit.

(a) [3] $a_n = \dfrac{3n}{e^{(3/n)}}$

(b) [3] $a_n = \dfrac{n \sin n}{(n^2+1)}$

14. [4] Write out the first five terms of the Maclaurin series of the function $\dfrac{1}{(1+x)^3}$.

Continued on Page 10. . .

15. Consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{3n+4}$.

   (a) [4] Show that the series is convergent.

   (b) [3] If S is the sum of the series, provide an estimate for the difference between S and S_{99}, the 99th partial sum of the series. Do not calculate S_{99}.

   (c) [3] Is the series absolutely convergent? Justify your answer to receive credit.

Continued on Page 11...

16. Consider the power series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n(n+1)}$.

(a) [6] Determine the radius of convergence of the power series.

(b) [3] Determine the interval of convergence of the power series.

Continued on Page 12. . .

17. (a) [3] Give the Maclaurin series of the function $f(x) = \cos x$.

(b) [3] Find the Maclaurin series for the function $g(x) = \cos(x^2)$. Hint: Use your solution from (a).

(c) [2] Evaluate $\int \cos(x^2)\, dx$ as an infinite series.

Continued on Page 13...

Arts & Science 1D06 Test #2

Day Class
Test #2
Duration of test: 60 minutes
McMaster University
28 February, 2012

Dr. Matt Valeriote

Last Name:______________________________ Initials:__________

Student No.:______________________________ Your TA's Name:______________

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on BOTH sides of the paper. Pages 7 and 8 contain no questions
and can be used for rough work.

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be accepted for regrading (if needed).

Good Luck!

Score

Question	1	2	3	4	5	6	7	Total
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Score								

continued . . .

ALL QUESTIONS: you must show your work to receive full credit.

1.(a) [4] State both parts of the Fundamental Theorem of Calculus

(b) [3] Use the Fundamental Theorem of Calculus to find the derivative of the function

$$y = \int_{\sin x}^{\cos x} (1 + v^2)^{10} \, dv.$$

continued . . .

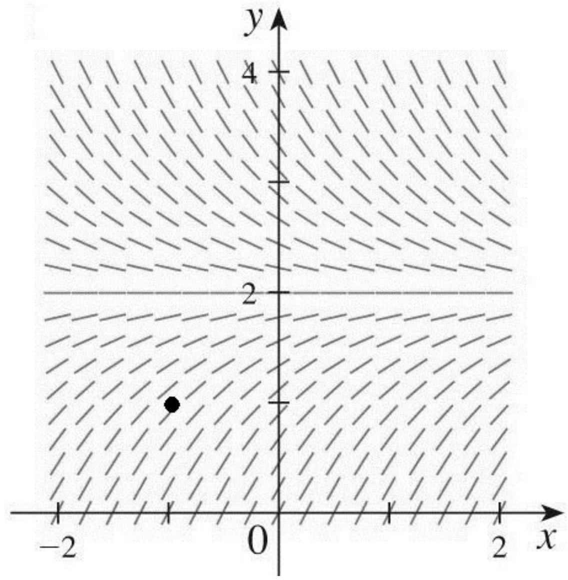
2. [6] Find $\int \sin\sqrt{x}\,dx$. [Hint: After making a suitable substitution, use integration by parts.]

3. [5] Let $p > 1$. Show that the following improper integral is convergent and evaluate it.

$$\int_1^\infty \frac{1}{x^p}\,dx.$$

continued . . .

4. [4] A direction field for a particular differential equation is given below.



(a) On the direction field, sketch the graph of the solution to the differential equation that passes through the point $(-1, 1)$.

(b) Which of the following differential equations matches the given direction field? Circle your answer.

(A) $\dfrac{dy}{dx} = 2 - y$ (B) $\dfrac{dy}{dx} = x^2 - y^2$ (C) $\dfrac{dy}{dx} = y - 1$ (D) $\dfrac{dy}{dx} = x^2 + 2$

continued . . .

5. [6] Let R be the region bounded by $x = 0$, $y = 1$, and the curve $y = \frac{x^3}{8}$. Sketch the region R and set up an integral for the volume of the solid obtained by rotating R about the horizontal line $y = -2$. **DO NOT EVALUATE THE INTEGRAL!**

6. [6] Solve the differential equation: $\frac{dy}{dx} = \frac{xy + 3x}{x^2 + 1}$.

continued …

7. [6] Let $P(t)$ be the population of the Earth t years after the year 2000 and assume that $P(t)$ grows according to the logistic equation

$$\frac{dP}{dt} = (0.02)P\left(1 - \frac{P}{60}\right).$$

In the year 2000 the population of the Earth was 6 billion people (so $P(0) = 6$).

(a) Write a formula for the population $P(t)$. You may use that the general solution of the logistic equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ is $\dfrac{M}{1 + Ae^{-kt}}$ for some constant A.

(b) What is the projected population of the Earth in the year 2100?

continued . . .

Name ______________________________

Student Number ______________________________

Your TA's Name: ______________________________

Arts & Science 1D06

DAY CLASS
APRIL EXAM
DURATION OF EXAM: 3 Hours
MCMASTER UNIVERSITY

DR. MATT VALERIOTE

10 April, 2012

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Score							

Continued on Page 2 . . .

Multiple Choice Questions

Indicate your answers to questions 1–3 by circling only ONE of the letters. Each of these questions is worth 3 marks.

1. [3] Let

$$f(x) = \frac{1}{e^x + 1}.$$

Which of the following statements are **true**?

(I) The domain of $f(x)$ is $(-\infty, \infty)$.

(II) $f(x)$ is an odd function.

(III) $f(x)$ has an inverse.

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

2. [3] Which of the following series are convergent?

$$\text{(I)} \quad \sum_{n=1}^{\infty} (-1)^n \qquad \text{(II)} \quad \sum_{n=1}^{\infty} 2^n \qquad \text{(III)} \quad \sum_{n=1}^{\infty} \frac{1}{2+n^3}$$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

Continued on Page 3...

3. [3] Let $g(x) = \int_{x^2}^{1} \sin(\sqrt{t})\, dt$. Then $g'(\pi/2)$ is:

(A) 0 (B) $-\pi$

(C) $\sin(1) - 1$ (D) -1

True/False Questions.

Decide whether the statements in questions 4–6 are true or false by circling your choice. YOU MUST JUSTIFY YOUR ANSWER TO RECEIVE FULL CREDIT. Each of these questions is worth 3 marks.

4. [3] There is some value of x in the interval $(2, 3)$ such that $x^3 - 5x - 7 = 0$.

TRUE FALSE

Continued on Page 4. . .

5. [3] The improper integral $\int_0^\infty \cos(x)\,dx$ is convergent.

TRUE FALSE

6. [3] If the power series $\sum_{n=1}^{\infty} c_n x^n$ converges when $x = -8$ then the series $\sum_{n=1}^{\infty} c_n 7^n$ converges.

TRUE FALSE

Questions 7–17: you must show work to receive full credit

7. [4] Sketch the region bounded by the curves $y = x^2$ and $y = 2x + 3$ and set up an integral for its area. **Do not evaluate the integral!**

Continued on Page 5 . . .

8.(a) [5] Let $f(x) = \dfrac{x}{1+x^2} + 1$. Find the absolute maximum and absolute minimum values of $f(x)$ on the interval $[-3, 2]$.

(b) [5] Find the intervals where $f(x) = \sin(2x) - 4\sin(x)$, $0 \le x \le \pi$, is concave up and concave down and identify all points of inflection.

Continued on Page 6...

9. Compute the following integrals.

   (a) [4] $\int_0^1 (x+1)e^{-x}\,dx$.

   (b) [4] $\int \frac{1}{(x+2)(x+3)}\,dx$.

Continued on Page 7...

10. [6] Consider the predator-prey system $x' = 4x - xy$, $y' = -y + \frac{xy}{2}$.

(a) Which of the variables, x or y, represents the predator? Explain why.

(b) For each of the species represented by x and y, explain what happens if the other is not present.

(c) Find all equilibrium solutions of this system.

11.[7] Find the solution of the differential equation $\frac{dy}{dx} = \frac{(x^2 - x)}{e^y}$, that satisfies the initial condition $y(0) = 1$.

Continued on Page 8...

12. [6] Suppose that the bowl of candy in C-105 initially contains 100 pieces and let $y = y(t)$ stand for the number of pieces of candy in the bowl after t hours.

(a) Find an exact expression for $y(t)$ assuming that one-third of the pieces in the bowl are removed each hour, and so y satisfies the differential equation $\dfrac{dy}{dt} = -\dfrac{y}{3}$.

(b) Now assume that Shelley is also continuously re-supplying the bowl at a rate of $\dfrac{75}{y}$ pieces per hour. Write a new differential equation that y satisfies in this case.

(c) Find the equilibrium amount of candy in the bowl in this situation.

Continued on Page 9...

13. Compute the following limits, or show that they do not exist. Justify your answers.

(a) [3] $\lim_{n\to\infty} \frac{e^n}{n!}$

(b) [3] $\lim_{n\to\infty} \frac{\ln(3n)}{\ln(n)}$

(c) [4] $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right)$.

Continued on Page 10...

14.(a) [2] Define the term "absolute convergence" for a series $\sum_{n=1}^{\infty} a_n$.

(b) [3] Give an example of a series that is convergent, but not absolutely convergent.

(c) [3] Determine if the series $\sum_{n=1}^{\infty} \frac{(-7)^n}{n6^n}$ is absolutely convergent.

Continued on Page 11...

15. Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2 2^n}$.

   (a) [4] Determine the radius of convergence of the power series.

   (b) [4] Determine the interval of convergence of the power series.

16. [5] Find the first four terms of the Maclaurin series for $f(x) = \dfrac{1}{\sqrt{1+2x}}$.

Continued on Page 12...

17. (a) [3] State the Maclaurin series of the function $f(x) = \cos x$. You do not need to derive the series.

(b) [3]Find the Maclaurin series for the function $g(x) = \dfrac{1-\cos(x)}{x^2}$. Hint: Use your answer from (a).

(c) [2] Use your answer from (b) to express $\int_0^1 \dfrac{1-\cos(x)}{x^2}\,dx$ as the sum of a series.

(d) [2] The sum of the first two terms of the series from (c) provides an approximation of the definite integral from (c). Give an estimate for the error of this approximation.

Continued on Page 13...