Arts & Science 1D6 Test #2

Day Class
Test #2
Duration of test: 60 minutes
McMaster University
1 March, 2011

Dr. Matt Valeriote

Last Name: Solutions   Initials:__________

Student No.:______________________________   Your TA's Name:____________

This test has 8 pages and 7 questions and is printed on BOTH sides of the paper. Pages 7 and 8 contain no questions and can be used for rough work.

You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancies to the attention of the invigilator.

Attempt all questions and write your answers in the space provided.

Marks are indicated next to each question; the total number of marks is 40.

Any Casio fx991 calculator is allowed. Other aids are not permitted.

Use pen to write your test. If you use a pencil, your test will not be accepted for regrading (if needed).

Good Luck!

Score

Question	1	2	3	4	5	6	7	Total
Points	6	5	6	6	6	6	5	40
Score								

continued ...

ALL QUESTIONS: you must show your work to receive full credit.

1. [6] Let $g(x) = \int_0^{x^2} te^{-t}\,dt$.

   (a) Compute $g'(x)$.

Use the Chain Rule & the F.T.C.:

$$g'(x) = (x^2)e^{-(x^2)} \cdot (x^2)'$$
$$= 2x^3 e^{-x^2}$$

   (b) Find the interval(s) on which the function $g(x)$ is concave upward

Compute $g''(x)$:

$$g''(x) = 6x^2e^{-x^2} + (2x^3)e^{-x^2}(-2x)$$
$$= 6x^2e^{-x^2} - 4x^4e^{-x^2}$$
$$= 2x^2e^{-x^2}\left[3-2x^2\right]$$

$g''(x) > 0$ when $3-2x^2 > 0$ or $x^2 < \frac{3}{2} \iff -\sqrt{\frac{3}{2}} < x < \sqrt{\frac{3}{2}}$

So $g(x)$ is concave upward on the interval $\left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$.

2. [5] Determine if the following improper integral is convergent and evaluate it if it is.

$$\int_0^2 \frac{x}{4-x^2}\,dx.$$

Since $\frac{x}{4-x^2}$ has a vertical asymptote at $x=2$, then this integral is improper.

$$\int_0^2 \frac{x}{4-x^2}dx = \lim_{t\to 2^-}\int_0^t \frac{x}{4-x^2}dx. \quad \text{Let } u = 4-x^2. \text{ Then } du = -2x\,dx$$

$$= \lim_{t\to 2^-}\int_4^{4-t^2} \frac{-1}{2u}du = \lim_{t\to 2^-} -\frac{1}{2}\ln|u|\Big|_4^{4-t^2}$$

$$= \lim_{t\to 2^-} -\frac{1}{2}\left(\ln|4-t^2| - \ln|4|\right) = -\frac{1}{2}\lim_{t\to 2^-}\ln|4-t^2| + \frac{\ln 4}{2}$$

$$= \infty, \text{ since } \lim_{t\to 2^-}\ln|4-t^2| = -\infty$$

So, this integral is divergent.

continued . . .

3. [6] Evaluate the following indefinite integrals.

(a) $\int \frac{e^{2x}}{1+e^{2x}}\,dx$ Use the substitution $u = 1+e^{2x}$

$du = 2e^{2x}dx$

$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|1+e^{2x}| + C = \frac{1}{2}\ln(1+e^{2x}) + C$ (since $1+e^{2x} > 0$)

(b) $\int 4x\cos(4x)\,dx$ Integrate by parts: $u = x$ $dv = 4\cos(4x)dx$

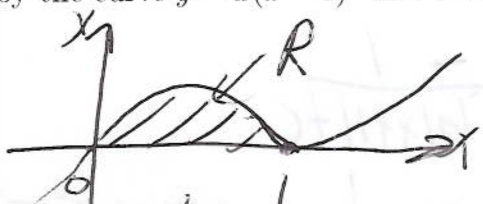
$du = dx$ $v = \sin(4x)$

$= (x)(\sin(4x)) - \int \sin(4x)dx$

$= x\sin(4x) + \frac{1}{4}\cos(4x) + C$

4. [6] Let R be the region bounded by the curve $y = x(x-1)^2$ and the x-axis.

(a) Find the area of R.



Area of $R = \int_0^1 x(x-1)^2 dx = \int_0^1 (x^3 - 2x^2 + x)dx$

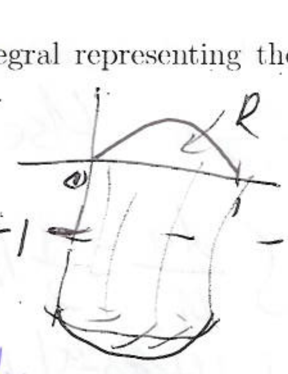
$= \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{x^2}{2}\right)\Big|_0^1$

$= \frac{1}{4} - \frac{2}{3} + \frac{1}{2} = \frac{3}{12} - \frac{8}{12} + \frac{6}{12}$

$= \frac{1}{12}$

continued ...

(b) Set up but DO NOT EVALUATE the integral representing the volume obtained by rotating the region R about the line $y = -1$.



Volume

$= \int_0^1 \pi(1+x(x-1)^2)^2 dx - \int_0^1 \pi(1)^2 dx$

$= \pi\int_0^1 (x^3-2x^2+x+1)^2 dx - \int_0^1 \pi dx$

$= \pi\left[\int_0^1 (x^3-2x^2+x+1)^2 dx - 1\right]$

5. [6] Solve the initial value problem $(x+1)\dfrac{dy}{dx} = y^2$, $y(0) = \frac{1}{2}$ for $x > -1$.

This is a separable D.E.

$\dfrac{dy}{y^2} = \dfrac{dx}{x+1}$

$\Rightarrow \int \dfrac{dy}{y^2} = \int \dfrac{dx}{x+1}$

$\Rightarrow -\dfrac{1}{y} = \ln|x+1| + C$

$\Rightarrow y = \dfrac{-1}{\ln|x+1|+C}$

or $y = \dfrac{-1}{\ln(x+1)+C}$ since $x > -1$

Since $y(0) = \frac{1}{2}$, then $\frac{1}{2} = \dfrac{-1}{\ln(0+1)+C} = -\dfrac{1}{C} \Rightarrow C = -2$

Thus $y = \dfrac{-1}{\ln(x+1)-2}$ or $\boxed{y = \dfrac{1}{2-\ln(x+1)}}$

continued ...

6. [6] A cup of coffee is poured from a pot whose contents are 95°C into a non-insulated cup in a room at 20°C. Let $T(t)$ be the temperature of the coffee after t minutes. Assuming that the coffee cools according to Newton's Law, then

$$\frac{dT}{dt} = k(T - 20).$$

(a) Solve this differential equation subject to the initial condition $T(0) = 95$.

This is a separable D.E.!

$\frac{dT}{T-20} = k\,dt$ if $T \neq 20$

$\Rightarrow \int \frac{dT}{T-20} = \int k\,dt$

$\Rightarrow \ln|T-20| = kt + C$

$\Rightarrow \ln(T-20) = kt + C$, since we can assume that $T-20>0$

$\Rightarrow T-20 = e^{C} \cdot e^{kt}$

$\Rightarrow T = e^{C}e^{kt} + 20$

Since $T(0)=95$, then $95 = e^{C}e^{k(0)} + 20 \Rightarrow e^{C} = 75$

Thus $T = 75e^{kt} + 20$

(b) After one minute, the coffee has cooled to 90°C. Use this information and your solution to (a) to solve for the constant k.

$T(t) = 75e^{kt} + 20$

$T(1) = 90 = 75e^{k} + 20$

$\Rightarrow 70 = 75e^{k}$

$\Rightarrow \frac{70}{75} = e^{k}$

so $\boxed{k = \ln\left(\frac{70}{75}\right)}$

continued . . .

7. [6] Populations of aphids (A) and ladybugs (L) are modeled by the predator-prey equations

$$\frac{dA}{dt} = 2A - 0.01AL$$
$$\frac{dL}{dt} = -0.5L + 0.0001AL$$

(a) Find the equilibrium solutions and explain their significance.

Set $\frac{dA}{dt}=0$ & $\frac{dL}{dt}=0$.

$2A-0.01AL=0$
$\Rightarrow A(2-0.01L)=0$
$\Rightarrow A=0$ or
$L=200$

$-0.5L+0.0001AL=0$
$\Rightarrow L(-0.5+0.0001A)=0$
$\Rightarrow L=0$ or $A=5000$

The equilibria solutions are $A=L=0$ or $A=5000, L=200$. At these values, the populations will remain unchanged over time.

(b) When there are 1000 aphids and 300 ladybugs, is the aphid population increasing or decreasing? Justify your answer.

$$\frac{dA}{dt} = 2A-0.01AL$$
$$=2(1000)-0.01(1000)(300)$$
$$=2000-3000$$
$$=-1000$$
$$<0$$

So, the population will be decreasing since $\frac{dA}{dt}$ under these conditions.

continued ...

Name Solutions

Student Number ______________________

Your TA's Name: ______________________

Arts & Science 1D6

DAY CLASS
DECEMBER EXAM
DURATION OF EXAM: 2 Hours
MCMASTER UNIVERSITY

DR. MATT VALERIOTE

17 December, 2011

THIS EXAMINATION PAPER INCLUDES 12 PAGES AND 12 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF YOUR INVIGILATOR.

Attempt all questions.
The total number of available points is 50.
Marks are indicated next to each question.
Use of a Casio fx991 calculator only is allowed.
Write your answers in the space provided.
You must show your work to get full credit.
Use the last page for rough work.
Good Luck.

Score

Question	1–4	5	6	7	8
Points	8	6	6	4	3
Score					

Question	9	10	11	12	Total
Points	6	3	9	5	50
Score					

Continued on Page 2 ...

Multiple Choice & True/False Questions.

Indicate your answers to questions 1 and 2 by circling only ONE of the letters. You do not need to provide justifications for your answers to these two questions. Each of these questions is worth 2 marks.

1. [2] If $g(x) = (f(x))^3 + f(x^3)$, $f(1) = 2$, and $f'(1) = -1$, then $g'(1)$ is equal to:

(A) 0 (B) -4 (C) 4 (D) -12

(E) 12 (F) -24 (G) 24 (H) -15

$$g'(x) = [(f(x))^3]' + (f(x^3))'$$
$$= 3(f(x))^2 \cdot f'(x) + f'(x^3)(3x^2) \quad [\text{Chain Rule}]$$

So, $g'(1) = 3(f(1))^2 \cdot f'(1) + f'(1^3)(3(1)^2)$
$$= 3(2)^2 \cdot (-1) + (-1)(3)$$
$$= -12 - 3 = \boxed{-15}$$

2. [2] Find the constant(s) c that make(s) the following function continuous everywhere:

$$f(x) = \begin{cases} c^2 - x^2 & \text{if } x < 2 \\ 2(c - x) & \text{if } x \geq 2 \end{cases}.$$

(A) $-4, -2$ (B) 0, 2 (C) 2 (D) 4

(E) $-2, 4$ (F) -2 (G) 0 (H) Does not exist

$c^2 - x^2$ & $2(c-x)$ are continuous functions for all constants c, so f will be continuous precisely when $c^2 - (2)^2 = 2(c-(2))$

or when: $c^2 - 4 = 2c - 4$
$$\Rightarrow c^2 = 2c$$
$$\Rightarrow c^2 - 2c = 0 \text{ or } c(c-2) = 0$$
$$\Rightarrow c = 0 \text{ or } c = 2$$

Continued on Page 3...

Indicate your answers to questions 3 and 4 by circling only ONE of TRUE or FALSE. To receive credit for your solutions, you must justify your answers. Each of these questions is worth 2 marks.

3. [2] If c is a critical number of the function f, then $f'(c) = 0$.

TRUE FALSE

f could have a critical number at c, if $f'(c)$ fails to exist. For example $c=0$ is a critical number of the function $f(x)=|x|$, but $f'(0) \neq 0$, since $f'(0)$ does not exist.

4. [2] If $g(x)$ is an even function that is continuous at all values, then $\int_{-1}^{1} xg(x)\,dx = 0$.

TRUE FALSE

Since $g(x)$ is an even function, then $f(x)=xg(x)$ is an odd function [proof: $f(-x)=(-x)g(-x) = -xg(x) = -f(x)$].

In general, for an odd function f, $\int_{-a}^{a} f(x)\,dx = 0$, since the area of the region bounded by $f(x)$ above the x-axis, for $x \geq 0$ is offset by the area of the region bounded by $f(x)$ below the x-axis, for $x \leq 0$.

Continued on Page 4 ...

Questions 5–12: you must show work to receive full credit.

5. Let $f(x) = \ln\left(\frac{x}{x-3}\right)$.

(a) [3] What is the domain of $f(x)$?

f(x) is defined when ① $\frac{x}{x-3}$ is defined & when

② $\frac{x}{x-3} > 0$.

So, f(x) is defined when $x \neq 3$ & [when ($x>0$ & $x-3>0$) or when ($x<0$ & $x-3<0$)]

So, the domain of f is $\{x \mid x>3 \text{ or } x<0\}$

$= (-\infty, 0) \cup (3, \infty)$

(b) [3] Find $f^{-1}(x)$, the inverse of the function $f(x)$. What is the domain of $f^{-1}(x)$ (and hence the range of $f(x)$)? [You do not need to show that $f(x)$ is one-to-one.]

① Set $y = f(x)$ & solve for x in terms of y:

$y = \ln\left(\frac{x}{x-3}\right)$

$e^y = \frac{x}{x-3}$

$(x-3)e^y = x$

$x(e^y - 1) = 3e^y$

$x = \frac{3e^y}{e^y - 1}$

② Interchange x & y:

$y = \frac{3e^x}{e^x - 1}$

③ $f^{-1}(x) = \frac{3e^x}{e^x - 1}$

The domain of $f^{-1}(x)$ ~~consi~~ consists of all x for which $e^x - 1 \neq 0$ or $e^x \neq 1$. Since $e^x = 1$ if and only if $x = 0$, then the domain of f^{-1} is $\{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$.

Continued on Page 5 ...

6. Compute the following limits:

(a) [2] $\lim_{x\to\infty} \frac{1+x\sqrt{x}}{\sqrt{x}+\frac{1}{\sqrt{x}}}$. ← Indeterminate of form $\frac{\infty}{\infty}$.

Divide numerator & denominator by $\sqrt{x}$, the highest power of x in the denominator:

$\lim_{x\to\infty} \frac{1+x\sqrt{x}}{\sqrt{x}+\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}(1+x\sqrt{x})}{\frac{1}{\sqrt{x}}(\sqrt{x}+\frac{1}{\sqrt{x}})} = \lim_{x\to\infty} \frac{(\frac{1}{\sqrt{x}}+x)}{1+\frac{1}{x}} = \infty.$

Note: L'Hospital's Rule could also be used to solve this limit.

(b) [2] $\lim_{h\to 0} \frac{e^{5+2h}-e^5}{h}$.

indeterminate of form $\frac{0}{0}$

$\overset{LH}{=} \lim_{h\to 0} \frac{(2)e^{5+2h}-0}{1} = \lim_{h\to 0} 2e^{5+2h} = 2\cdot e^5$

Note: If $f(x) = e^{5+2x}$, then $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{e^{5+2h}-e^5}{h}$.

Since $f'(x) = 2\cdot e^{5+2x}$, then $f'(0) = 2\cdot e^{5+2(0)} = 2e^5$.

So, $\lim_{h\to 0} \frac{e^{5+2h}-e^5}{h} = 2e^5$.

(c) [2] $\lim_{x\to 0^+} x^{(1/x)}$. ← this limit has form 0^∞, which is determinate.

As $x\to 0^+$, $x^{(\frac{1}{x})}$ approaches 0 & so

$\lim_{x\to 0^+} x^{(1/x)} = 0.$

⇒ Alternatively: Since $\lim_{x\to 0^+} (\frac{1}{x})\ln(x) = -\infty$, then

$\lim_{x\to 0^+} x^{(1/x)} = \lim_{x\to 0^+} e^{[\frac{1}{x}\ln(x)]} = e^{[\lim_{x\to 0^+} (\frac{1}{x})\ln x]} = e^{-\infty} = 0.$

Continued on Page 6 . . .

7. It is estimated that following the major earthquake that struck off of the coast of Japan earlier this year, 2,000 grams of the radioactive substance Cesium-137 were released into the atmosphere from the crippled Fukushima Daiichi Nuclear Power Plant. The half-life of Cesium-137 is 30 years, and so 30 years from now, one-half of the released material will remain.

   (a) [2] Find a formula for $m(t)$, the mass of the remaining Cesium-137, after t years.

$m(t) = Ae^{kt}$ for some constants A & k.
$m(0) = Ae^{k(0)} = A$. We are given that $m(0) = 2000$ grams
So $A = 2000$.
Also, $m(30) = \frac{1}{2}m(0) = \frac{1}{2}(2000) = 1000$
so $1000 = 2000e^{k(30)}$
$\Rightarrow e^{30k} = \frac{1}{2} \Rightarrow 30k = \ln(\frac{1}{2}) = -\ln(2)$

$\Rightarrow k = -\frac{\ln(2)}{30}$.
So, $m(t) = 2000e^{-\frac{\ln(2)}{30}t}$ grams
$\boxed{= 2000 \cdot 2^{-\frac{t}{30}}}$ grams

   (b) [2] How long will it take for the mass of the remaining Cesium-137 to be reduced to 100 grams?

Solve for t in: $m(t) = 100$: $e^{-\frac{\ln(2)}{30}t} = \frac{1}{20}$ or $e^{\frac{\ln(2)}{30}t} = 20$
$100 = 2000e^{\left(-\frac{\ln(2)}{30}t\right)} \Rightarrow$
$\Rightarrow \frac{\ln(2)}{30}t = \ln(20)$ or ~~$\frac{600}{\ln(2)}$ years~~
$t = \frac{30 \cdot \ln(20)}{\ln(2)} = 129.66$ years

8. [3] Let $g(x)$ be a function that is continuous on the interval $[2, 4]$. If $g(2) > 2$ and $g(4) < 4$ show that there is a solution to the equation $g(x) = x$ in the interval $(2, 4)$, i.e., there is some number c with $2 < c < 4$ and with $g(c) = c$.

Use the Intermediate Value Theorem!
Let $f(x) = g(x) - x$ & show there is some number c with $2 < c < 4$ with $f(c) = 0$.
Since $g(2) > 2$, then $g(2) - 2 > 0$, so $f(2) > 0$.
Since $g(4) < 4$, then $g(4) - 4 < 0$, so $f(4) < 0$.
Since g is continuous on $[2,4]$, then so is f &
thus by the IVT, there is some c in $(2,4)$
with $f(c) = 0$, or with $g(c) = c$.

Continued on Page 7 ...

9. Find the derivatives of the following functions. You do not need to simplify your answers.

(a) [3] $h(t) = \arcsin(t^2) - \frac{e^t}{1+e^{2t}}$.

$$h'(t) = \frac{1}{\sqrt{1-(t^2)^2}} \cdot (2t) - \frac{e^t(1+e^{2t}) - e^t(2e^{2t})}{(1+e^{2t})^2}$$ [Chain Rule & Quotient Rule]

$$= \frac{2t}{\sqrt{1-t^4}} - \frac{e^t - e^{3t}}{(1+e^{2t})^2}$$

(b) [3] $f(x) = \ln(x)\cos(\tan(x))$.

$$f'(x) = \frac{1}{x} \cdot \cos(\tan(x)) + \ln(x) \cdot (-\sin(\tan(x)) \cdot \sec^2(x))$$

$$= \frac{\cos(\tan(x))}{x} - \ln(x)\sin(\tan(x))\sec^2(x).$$

10. [3] Find the function $f(x)$ that satisfies the given conditions:

$$f'(x) = (x-1)^3 + 2 + \frac{1}{1+x^2} \text{ and } f(1) = 2.$$

The most general anti-derivative of the given function is

$$f(x) = \frac{1}{4}(x-1)^4 + 2x + \arctan(x) + C$$

Using $f(1) = 2$, we can solve for C:

$$f(1) = \frac{1}{4}(1-1)^4 + 2(1) + \arctan(1) + C = 2$$

$$\Rightarrow 2 + \arctan(1) + C = 2$$

$$\Rightarrow C = -\arctan(1) = -\frac{\pi}{4}$$

So, $f(x) = \frac{1}{4}(x-1)^4 + 2x + \arctan(x) - \frac{\pi}{4}$

Continued on Page 8 ...

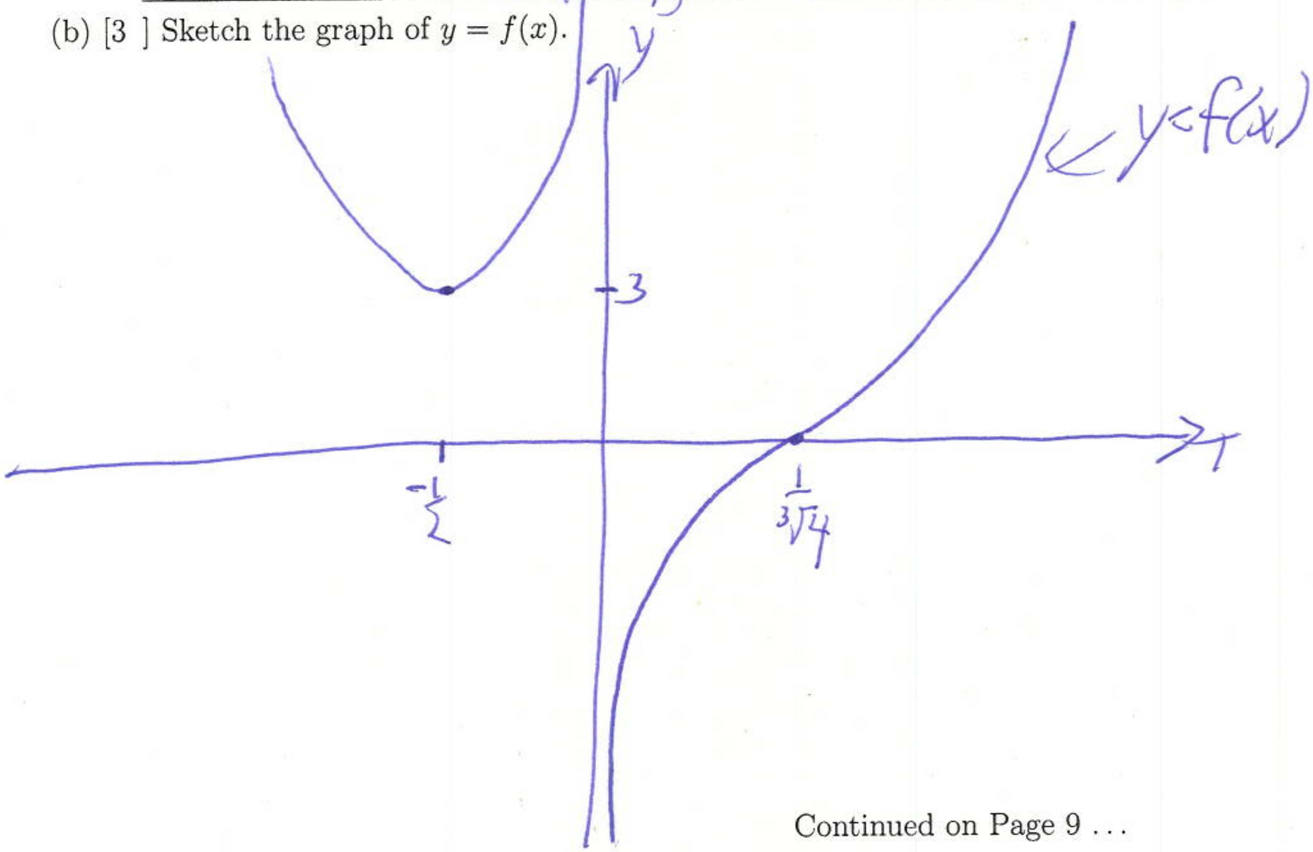
11. Let $f(x) = 4x^2 - \dfrac{1}{x}$; then $f'(x) = 8x + \dfrac{1}{x^2}$ and $f''(x) = 8 - \dfrac{2}{x^3}$.

(a) [6] For the function f, find the domain, $x-$ and $y-$ intercepts, any symmetries, all asymptotes, all local extreme values and intervals of increase and decrease, all intervals where f is concave up and concave down, and inflection points. Place your answers in the following table. Use the next page for rough work.

ANSWERS:

domain of f:	$x \neq 0$
$x-$intercept(s):	$x = \frac{1}{\sqrt[3]{4}}$
$y-$intercept(s):	——
symmetries:	None
horizontal asymptote(s):	None
vertical asymptote(s):	$x=0$. $\lim_{x\to 0^+} f(x) = -\infty$, $\lim_{x\to 0^-} f(x) = +\infty$
local extreme values (if any):	local min at $x=-\frac{1}{2}$, $f(-\frac{1}{2}) = 3$
f is increasing on:	$x>0$ or $-\frac{1}{2}<x<0$
f is decreasing on:	$x<-\frac{1}{2}$.
inflection points (if any):	$x = \frac{1}{\sqrt[3]{4}}$
f is concave up on:	$x > \frac{1}{\sqrt[3]{4}}$ or $x<0$
f is concave down on:	$0<x<\frac{1}{\sqrt[3]{4}}$, $x<0$

(b) [3] Sketch the graph of $y = f(x)$.



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Space for rough work for question #11.

x-int: Solve $4x^2 - \frac{1}{x} = 0 \Rightarrow 4x^2 = \frac{1}{x} \Rightarrow 4x^3 = 1 \Rightarrow x = \frac{1}{\sqrt[3]{4}}$

$\lim_{x \to 0^+} f(x) = -\infty$, $\lim_{x \to 0^-} f(x) = \infty$

$f'(x) = 8x + \frac{1}{x^2}$

$f'(x) = 0$ when $8x + \frac{1}{x^2} = 0$ or $8x = -\frac{1}{x^2}$ or $8x^3 = -1$
$\Rightarrow x = -\frac{1}{2}$

$f'(x) > 0$ when $x > 0$ or when $-\frac{1}{2} < x < 0$
$f'(x) < 0$ when $x < -\frac{1}{2}$
So f has a local min at $x = -\frac{1}{2}$

$f''(x) = 0$ when $8 - \frac{2}{x^3} = 0$ or $8x^3 = 2$ or $x^3 = \frac{2}{8}$ or $x = \frac{1}{\sqrt[3]{4}}$

$f''(x) > 0$ when $8 - \frac{2}{x^3} > 0$ or $\underset{x>0}{8x^3 > 2}$ or $x^3 > \frac{1}{4}$ or $x > \frac{1}{\sqrt[3]{4}}$
or $x < 0$
$f''(x) < 0$ when $0 < x < \frac{1}{\sqrt[3]{4}}$

Continued on Page 10 ...

12. (a) [3] Estimate $\int_1^3 (x - x^2)\, dx$ by evaluating the Riemann sum for $f(x) = x - x^2$, with $n = 4$, and by taking the sample points to be the midpoints of each subinterval.

$a=1, b=3, n=4, \Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$

$x_0 = a = 1, x_1 = a+\Delta x = \frac{3}{2}, x_2 = a+2\Delta x = 1+1 = 2, x_3 = a+3\Delta x = 1+\frac{3}{2} = \frac{5}{2}, x_4 = b = 3$

$M_4 = \sum_{i=1}^{4} f(\bar{x}_i)\Delta x$, where $\bar{x}_i = \frac{x_{i-1}+x_i}{2}$

$\bar{x}_1 = \frac{1+\frac{3}{2}}{2} = \frac{5}{4}, \bar{x}_2 = \frac{\frac{3}{2}+2}{2} = \frac{7}{4}, \bar{x}_3 = \frac{2+\frac{5}{2}}{2} = \frac{9}{4}, \bar{x}_4 = \frac{\frac{5}{2}+3}{2} = \frac{11}{4}$

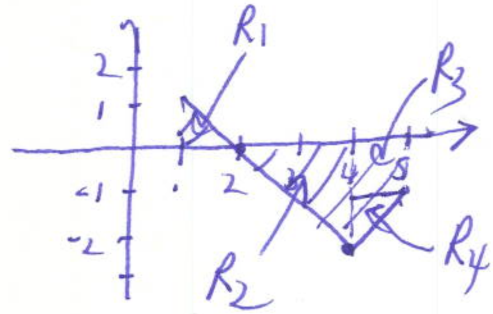
$M_4 = f(\frac{5}{4})(\frac{1}{2}) + f(\frac{7}{4})(\frac{1}{2}) + f(\frac{9}{4})(\frac{1}{2}) + f(\frac{11}{4})(\frac{1}{2})$

$= \frac{1}{2}\left[f(\frac{5}{4}) + f(\frac{7}{4}) + f(\frac{9}{4}) + f(\frac{11}{4})\right]$

$= \frac{1}{2}\left[(\frac{5}{4}+\frac{7}{4}+\frac{9}{4}+\frac{11}{4}) - ((\frac{5}{4})^2+(\frac{7}{4})^2+(\frac{9}{4})^2+(\frac{11}{4})^2)\right] = \frac{1}{2}\left[8 - \frac{276}{16}\right] = -\frac{37}{8} = -4.625.$

(b) [2] Find the exact value of the definite integral $\int_1^5 (|x-4| - 2)\, dx$. (Hint: Interpret the definite integral in terms of net area.)

The graph of $y = |x-4|-2$ is:



Area of region above the x-axis is $(1)(1)\cdot\frac{1}{2} = \frac{1}{2}$

Area of region below the x-axis is: $(2)(2)(\frac{1}{2}) + (1)(1) + 1\cdot 1\cdot(\frac{1}{2})$

$= 2+1+\frac{1}{2} = 3\frac{1}{2}$

Net area $= (\frac{1}{2}) - (3\frac{1}{2})$

$\boxed{= -3}$

Continued on Page 11 ...

Arts & Science 1D06 Test #2

Day Class
Test #2
Duration of test: 60 minutes
McMaster University
28 February, 2012

Dr. Matt Valeriote

Last Name: Solutions ______ Initials: ______

Student No.: ______ Your TA's Name: ______

This test has 8 pages and 7 questions and is printed on BOTH sides of the paper. Pages 7 and 8 contain no questions and can be used for rough work.

You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancies to the attention of the invigilator.

Attempt all questions and write your answers in the space provided.

Marks are indicated next to each question; the total number of marks is 40.

Any Casio fx991 calculator is allowed. Other aids are not permitted.

Use pen to write your test. If you use a pencil, your test will not be accepted for regrading (if needed).

Good Luck!

Score

Question	1	2	3	4	5	6	7	Total
Points	7	6	5	4	6	6	6	40
Score								

continued ...

ALL QUESTIONS: you must show your work to receive full credit.

1(a) [4] State both parts of the Fundamental Theorem of Calculus

Suppose that f is continuous on $[a,b]$.

(1) If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$

(2) $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, i.e. $F' = f$.

(b) [3] Use the Fundamental Theorem of Calculus to find the derivative of the function

$$y = \int_{\sin x}^{\cos x} (1+v^2)^{10}\,dv.$$

$\Rightarrow$ We also need to use the Chain Rule

$$y = \int_{\sin x}^{\cos x} (1+v^2)^{10}\,dv = \int_{\sin x}^{0} (1+v^2)^{10}\,dv + \int_0^{\cos x} (1+v^2)^{10}\,dv$$

$$= -\int_0^{\sin x} (1+v^2)^{10}\,dv + \int_0^{\cos x} (1+v^2)^{10}\,dv,$$

so $$y' = \left[-\int_0^{\sin x} (1+v^2)^{10}\,dv\right]' + \left[\int_0^{\cos x} (1+v^2)^{10}\,dv\right]'$$

$$= -(1+\sin^3 x)^{10}(\sin x)' + (1+\cos^2 x)^{10}(\cos x)'$$

$$= -(1+\sin^3 x)^{10}(\cos x) - (1+\cos^3 x)^{10}(\sin x).$$

continued ...

2. [6] Find $\int \sin\sqrt{x}\,dx$. [Hint: After making a suitable substitution, use integration by parts.]

Let $s=\sqrt{x}$. Then $ds = \frac{1}{2\sqrt{x}}dx = \frac{1}{2s}dx$

so $\int \sin\sqrt{x}dx = \int \sin(s)\cdot 2s\,ds = \int 2s\cdot\sin(s)ds.$

Use Int. by Parts with $u=2s$, $dv=\sin(s)ds$
$du=2ds$ $v=-\cos(s)$

$$\text{so } \int 2s\cdot\sin(s)ds = uv-\int v\,du = (2s)(-\cos(s)) - \int(-\cos(s))\cdot 2ds$$
$$= -2s\cos(s) + 2\int\cos(s)ds$$
$$= -2s\cos(s) + 2\sin(s) + C$$
$$= -2\sqrt{x}\cdot\cos(\sqrt{x}) + 2\sin(\sqrt{x}) + C$$

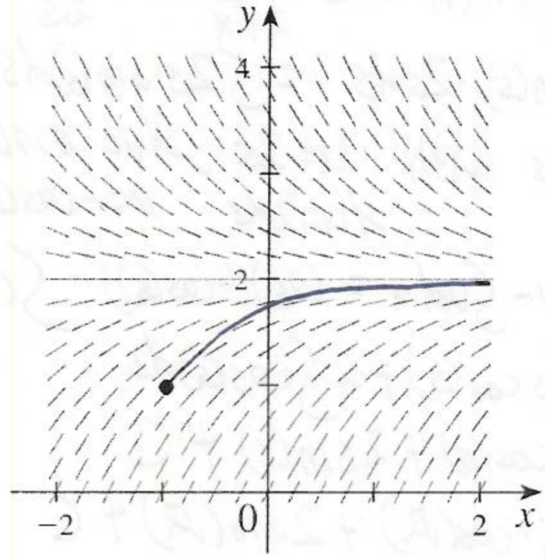
3. [5] Let $p>1$. Show that the following improper integral is convergent and evaluate it.

$$\int_1^\infty \frac{1}{x^p}\,dx.$$

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}dx$$
$$= \lim_{t\to\infty}\left[\frac{1}{-p+1}\cdot\frac{1}{x^{p-1}}\Big|_1^t\right]$$
$$= \lim_{t\to\infty}\left[\frac{1}{1-p}\cdot\frac{1}{t^{p-1}} - \frac{1}{1-p}\cdot\frac{1}{1^{p-1}}\right]$$
$$= \left(\frac{1}{1-p}\cdot\lim_{t\to\infty}\frac{1}{t^{p-1}}\right) - \frac{1}{1-p}$$
$$= 0 + \frac{1}{p-1}, \text{ since } \lim_{t\to\infty}\frac{1}{t^{p-1}} = 0 \text{ when } p>1$$
$$\boxed{= \frac{1}{p-1}}$$

continued . . .

4. [4] A direction field for a particular differential equation is given below.



(a) On the direction field, sketch the graph of the solution to the differential equation that passes through the point $(-1, 1)$.

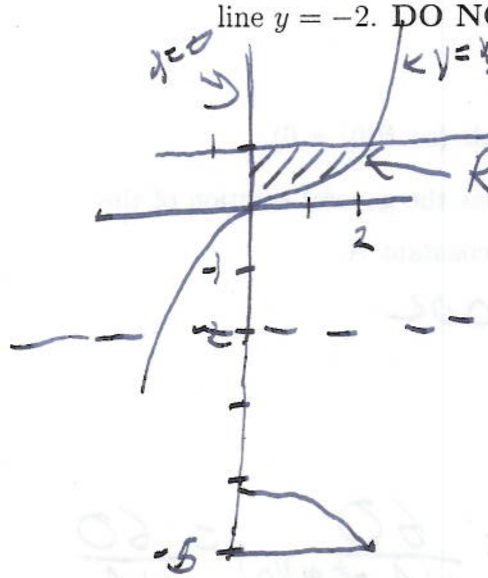
(b) Which of the following differential equations matches the given direction field? Circle your answer.

(A) $\frac{dy}{dx} = 2 - y$ (B) $\frac{dy}{dx} = x^2 - y^2$ (C) $\frac{dy}{dx} = y - 1$ (D) $\frac{dy}{dx} = x^2 + 2$

(*) Observe: that the slopes of the tangent lines in the Direction Field do not depend on the value of x. This rules out (B) & (D).
- Also when $y = 2$, the slope of the tangent lines are 0. This rules out (C) & so the answer is (A).

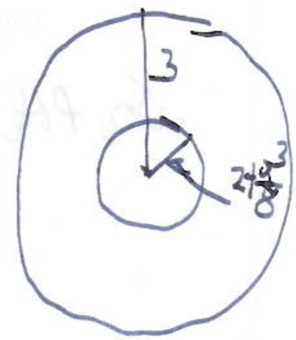
continued . . .

5. [6] Let R be the region bounded by $x = 0$, $y = 1$, and the curve $y = \frac{x^3}{8}$. Sketch the region R and set up an integral for the volume of the solid obtained by rotating R about the horizontal line $y = -2$. **DO NOT EVALUATE THE INTEGRAL!**



$$V = \int_0^2 \left[\pi(3)^2 - \pi\left(2+\frac{x^3}{8}\right)^2\right] dx$$

$$= \pi \int_0^2 \left[9 - \left(2+\frac{x^3}{8}\right)^2\right] dx$$



6. [6] Solve the differential equation: $\frac{dy}{dx} = \frac{xy + 3x}{x^2 + 1}$.

This is a seperable first order d.e.!

$$\frac{dy}{dx} = \frac{x(y+3)}{x^2+1} \quad (*)$$

$$\Rightarrow \frac{dy}{y+3} = \frac{x}{x^2+1}\,dx \quad \text{if } y+3 \neq 0, \text{ or } y \neq -3$$

$$\left[\text{let } u = x^2+1,\ du = 2x\,dx \quad \int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C\right]$$

$$\Rightarrow \int \frac{dy}{y+3} = \int \frac{x}{x^2+1}dx$$

$$\Rightarrow \ln|y+3| = \frac{1}{2}\ln(x^2+1) + C$$

$$\text{or } \ln|y+3| = \ln\left(\sqrt{x^2+1}\right) + C$$

$$\Rightarrow |y+3| = e^{\ln\sqrt{x^2+1}} \cdot e^C$$

$$\Rightarrow y+3 = \pm e^C \sqrt{x^2+1}$$

$$y = -3 \pm e^C \sqrt{x^2+1}$$

$$\Rightarrow y = -3 + A\sqrt{x^2+1}, \quad A \text{ any nonzero constant.}$$

(*) But: $y = -3$ is also a solution, since $\frac{dy}{dx} = 0 = \frac{x(-3)+3x}{x^2+1}$.

So, the general solution is $y = -3 + A\sqrt{x^2+1}$ for any constant A.

continued ...

7. [6] Let $P(t)$ be the population of the Earth t years after the year 2000 and assume that $P(t)$ grows according to the logistic equation

$$\frac{dP}{dt} = (0.02)P\left(1 - \frac{P}{60}\right).$$

In the year 2000 the population of the Earth was 6 billion people (so $P(0) = 6$).

(a) Write a formula for the population $P(t)$. You may use that the general solution of the logistic equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ is $\frac{M}{1 + Ae^{-kt}}$ for some constant A.

So, $P(t) = \frac{M}{1+Ae^{-kt}}$, where $M = 60$ & $k = 0.02$

$= \frac{60}{1+Ae^{-(0.02)t}}$

To solve for A: use $P(0) = 6$ & $P(0) = \frac{60}{1+Ae^{-(0.02)(0)}} = \frac{60}{1+A}$

$\Rightarrow 6 = \frac{60}{1+A} \Rightarrow 1+A = \frac{60}{6} = 10 \Rightarrow \underline{A=9}$ [or use: $A = \frac{M-P_0}{P_0}$]

Thus $P(t) = \frac{60}{1+9e^{-0.02t}}$

(b) What is the projected population of the Earth in the year 2100?

The population in the year 2100 is equal to

$P(100) = \frac{60}{1+9e^{-0.02(100)}} = \frac{60}{1+9(0.13533..)} = 27.05$

So, according to this model, the population of the Earth in 2100 will be 27 billion.

Name ____________________

Student Number ____________________

Your TA's Name: ____________________

Arts & Science 1D06

DAY CLASS
APRIL EXAM
DURATION OF EXAM: 3 Hours
MCMASTER UNIVERSITY

DR. MATT VALERIOTE

10 April, 2012

THIS EXAMINATION PAPER INCLUDES 14 PAGES AND 17 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF YOUR INVIGILATOR.

Attempt all questions.
The total number of available points is 100.
Marks are indicated next to each question.
Use of a Casio fx991 calculator only is allowed.
Write your answers in the space provided.
You must show your work to get full credit.
Use the last two pages for rough work.
Good Luck.

Score

Question	1–3	4–6	7	8	9	10	11
Points	9	9	4	10	8	6	7
Score							

Question	12	13	14	15	16	17	Total
Points	6	10	8	8	5	10	100
Score							

Continued on Page 2 ...

Multiple Choice Questions

Indicate your answers to questions 1–3 by circling only ONE of the letters. Each of these questions is worth 3 marks.

1. [3] Let

$$f(x) = \frac{1}{e^x + 1}.$$

Which of the following statements are **true**?

(I) The domain of $f(x)$ is $(-\infty, \infty)$. TRUE

(II) $f(x)$ is an odd function. FALSE

(III) $f(x)$ has an inverse. TRUE

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

$y = \frac{1}{e^x+1} \Rightarrow \frac{1}{y} = e^x+1 \Rightarrow e^x = \frac{1}{y} - 1 \Rightarrow x = \ln(\frac{1}{y}-1)$

So, $f^{-1}(x) = \ln(\frac{1}{x} - 1)$

2. [3] Which of the following series are convergent?

$$\text{(I)} \sum_{n=1}^{\infty} (-1)^n \quad \text{(II)} \sum_{n=1}^{\infty} 2^n \quad \text{(III)} \sum_{n=1}^{\infty} \frac{1}{2+n^3}$$

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

$\sum_{n=1}^{\infty} (-1)^n$ is divergent, since $\lim_{n\to\infty} (-1)^n \neq 0$ (Test for Divergence)

$\sum_{n=1}^{\infty} 2^n$ is divergent, since $\lim_{n\to\infty} 2^n \neq 0$ (Test for Divergence)

$\sum_{n=1}^{\infty} \frac{1}{2+n^3}$ is convergent, by comparison with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Continued on Page 3...

3. [3] Let $g(x) = \int_{x^2}^{1} \sin(\sqrt{t})\,dt$. Then $g'(\pi/2)$ is:

(A) 0 (B) $-\pi$

(C) $\sin(1) - 1$ (D) -1

$g(x) = -\int_1^{x^2} \sin(\sqrt{t})dt$, so by th FTC & Chain Rule,

$g'(x) = -\sin(\sqrt{x^2})(x^2)'$

$= -\sin(x)(2x)$ (if $x>0$)

$= -2x\sin(x)$

so $g'(\frac{\pi}{2}) = -2(\frac{\pi}{2})\sin(\frac{\pi}{2}) = -\pi(1) = -\pi$

True/False Questions.

Decide whether the statements in questions 4–6 are true or false by circling your choice. YOU MUST JUSTIFY YOUR ANSWER TO RECEIVE FULL CREDIT. Each of these questions is worth 3 marks.

4. [3] There is some value of x in the interval $(2, 3)$ such that $x^3 - 5x - 7 = 0$.

TRUE FALSE

Use the Intermediate Value Theorem:

Let $f(x) = x^3 - 5x - 7$.

Then f is continuous on $[2,3]$ since it is a polynomial,

$f(2) = 2^3 - 5(2) - 7 = 8 - 10 - 7 = -9 < 0$

& $f(3) = 3^3 - 5(3) - 7$

$= 27 - 15 - 7 = 5 > 0$

So, by the IVT there is some x with $2 < x < 3$

& $f(x) = 0$, i.e.,

$x^3 - 5x - 7 = 0$

Continued on Page 4...

5. [3] The improper integral $\int_0^\infty \cos(x)\,dx$ is convergent.

TRUE   FALSE

$\int_0^\infty \cos(x)\,dx = \lim_{t\to\infty} \int_0^t \cos(x)\,dx = \lim_{t\to\infty} \sin(x)\Big|_0^t = \lim_{t\to\infty}(\sin(t) - \sin(0))$

$= \lim_{t\to\infty} \sin(t) \leftarrow$ Does Not Exist.

So, the integral is divergent.

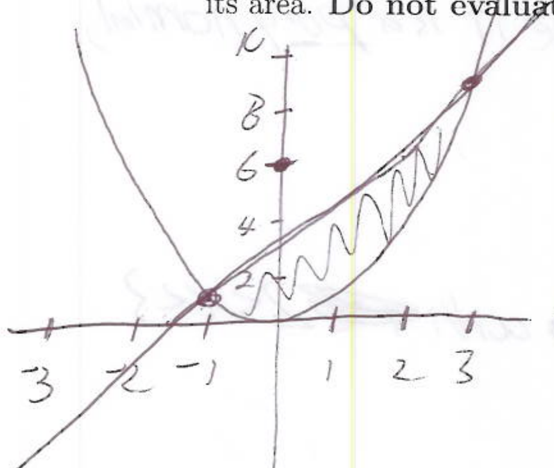
6. [3] If the power series $\sum_{n=1}^{\infty} c_n x^n$ converges when $x = -8$ then the series $\sum_{n=1}^{\infty} c_n 7^n$ converges.

TRUE   FALSE

Since the power series converges when $x=-8$, then R, the radius of convergence of the power series is ≥ 8. So, 7 is in the interval of convergence of the power series so $\sum_{n=1}^{\infty} c_n 7^n$ is convergent.

Questions 7–17: you must show work to receive full credit

7. [4] Sketch the region bounded by the curves $y = x^2$ and $y = 2x + 3$ and set up an integral for its area. **Do not evaluate the integral!**



To find the points of intersection, solve

$x^2 = 2x+3 \Rightarrow x^2-2x-3=0$

$\Rightarrow (x-3)(x+1)=0 \Rightarrow x=3$ or $x=-1$

$\text{Area} = \int_{-1}^{3} (2x+3) - x^2\,dx$

$= \int_{-1}^{3} (2x+3-x^2)\,dx$

Continued on Page 5 ...

8.(a) [5] Let $f(x) = \dfrac{x}{1+x^2} + 1$. Find the absolute maximum and absolute minimum values of $f(x)$ on the interval $[-3, 2]$.

Since f is continuous on $[-3,2]$, then by the Extreme Value Theorem, the absolute maximum & minimum will occur at the endpoints, -3 or 2, or at a critical point.

To find the critical points, solve $f'(x)=0$: $f'(x) = \dfrac{(1+x^2)-x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$.

So $f'(x)=0$ when $x^2=1$ or $x=\pm 1$.

Evaluate f at $-3, -1, 1, \& 2$: $f(-3) = \frac{-3}{10}+1 = 0.7$, $f(-1) = \frac{-1}{1+1^2}+1 = 0.5$

$f(1) = \frac{1}{1+1^2}+1 = 1.5$, $f(2) = \frac{2}{1+2^2}+1 = 1.4$

The largest value is 1.5 & so the absolute max. value of f on $[-3,2]$ is 1.5

The smallest value is 0.5 & so the absolute min value of f on $[-3,2]$ is 0.5.

(b) [5] Find the intervals where $f(x) = \sin(2x) - 4\sin(x)$, $0 \le x \le \pi$, is concave up and concave down and identify all points of inflection.

Find where $f''(x)$ is <0, >0, or $=0$:

$f'(x) = 2\cos(2x) - 4\cos(x)$, $f''(x) = -4\sin(2x) + 4\sin(x)$

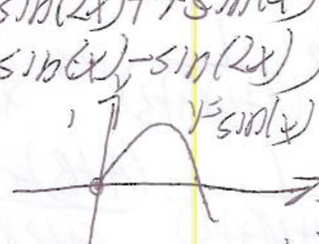
$= 4(\sin(x) - \sin(2x))$

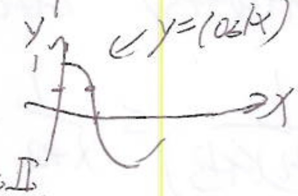
Solve $f''(x)=0$: $4(\sin(x)-\sin(2x))=0$

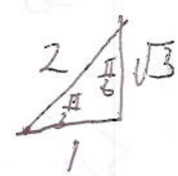
$\Rightarrow \sin(x) - 2\sin(x)\cos(x) = 0$

$\Rightarrow \sin(x)[1-2\cos(x)] = 0$

$\Rightarrow \sin(x)=0$, or $\cos(x) = \frac{1}{2}$





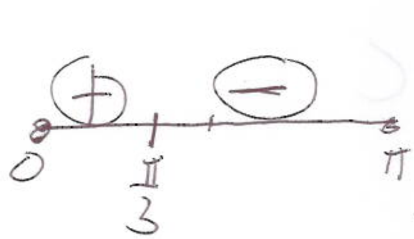


on $[0,\pi]$, this implies $x=0$ or $x=\pi$, or $x=\frac{\pi}{3}$.

So, on $[0,\pi]$, $f''(x)=0$ when $x=0, \frac{\pi}{3}$, or π.

In the interval $(0,\frac{\pi}{3})$, $f''(x) < 0$ (since $f''(\frac{\pi}{4})<0$)

& in the interval $(\frac{\pi}{3},\pi)$, $f''(x) > 0$ (since $f''(\frac{\pi}{2}) = 1$)



So f is Concave Down on $(0,\frac{\pi}{3})$ &

Concave up on $(\frac{\pi}{3},\pi)$.

f has points of inflection at $x=0, \frac{\pi}{3}$ & π.

Continued on Page 6...

9. Compute the following integrals.

(a) [4] $\int_0^1 (x+1)e^{-x}\,dx.$ Use Integration by Parts;

$u=(x+1)$, $dv=e^{-x}dx$

$du=dx$ $\quad v=-e^{-x}$

$\int_0^1 (x+1)e^{-x}dx = (x+1)(-e^{-x})\Big|_0^1 - \int_0^1 (-e^{-x})dx$

$=\left[(1+1)(-e^{-1})-(0+1)(-e^{0})\right]+(-e^{-x})\Big|_0^1$

$=\left(\frac{-2}{e}+1\right)+\left[-e^{-1}+e^{0}\right]$

$=\frac{-2}{e}+1-\frac{1}{e}+1$

$=2-\frac{3}{e}$

(b) [4] $\int \frac{1}{(x+2)(x+3)}\,dx.$ ← Use Partial Fractions Method

Write $\frac{1}{(x+2)(x+3)}$ as $\frac{A}{x+2}+\frac{B}{x+3}=\frac{A(x+3)+B(x+2)}{(x+2)(x+3)}=\frac{(A+B)x+(3A+2B)}{(x+2)(x+3)}$

So $\frac{1}{(x+2)(x+3)}=\frac{(A+B)x+(3A+2B)}{(x+2)(x+3)} \Rightarrow \begin{matrix} A+B=0 \\ 3A+2B=1\end{matrix} \Rightarrow B=-A \Rightarrow 3A-2A=1$

$\Rightarrow A=1$

$\Rightarrow \underline{B=-1}$

So $\frac{1}{(x+2)(x+3)}=\frac{1}{x+2}-\frac{1}{x+3}$

Thus $\int\frac{1}{(x+2)(x+3)}dx=\int\frac{1}{x+2}dx-\int\frac{1}{x+3}dx$

$=\ln|x+2|-\ln|x+3|+C$

$=\ln\left|\frac{x+2}{x+3}\right|+C$

Continued on Page 7...

10. [6] Consider the predator-prey system $x' = 4x - xy$, $y' = -y + \frac{xy}{2}$.

(a) Which of the variables, x or y, represents the predator? Explain why.

y represents the predator, since the term $+\frac{xy}{2}$ indicates that the y population will increase when there are interactions between the two species. The term $-xy$ in the first equation indicates that the x population will decrease when there are interactions.

(b) For each of the species represented by x and y, explain what happens if the other is not present.

when $y=0$, $x'=4x$ & so x will grow exponentially, $x(t)=x_0e^{4t}$.

When $x=0$, $y'=-y$ & so y will decrease exponentially & $y(t)=y_0e^{-t}$.

(c) Find all equilibrium solutions of this system.

Solve $x'=0$ & $y'=0$: $4x-xy=0 \Rightarrow x(4-y)=0 \Rightarrow x=0$ or $y=4$

& $-y+\frac{xy}{2}=0 \Rightarrow y(-1+\frac{x}{2})=0 \Rightarrow y=0$ or $x=2$

So, the equilibrium solutions are $x=0$ & $y=0$ & $x=2$ & $y=4$.

11.[7] Find the solution of the differential equation $\frac{dy}{dx} = \frac{(x^2-x)}{e^y}$, that satisfies the initial condition $y(0) = 1$.

This is a separable d.e.: $e^y dy = (x^2-x)dx$

$\Rightarrow \int e^y dy = \int (x^2-x)dx \Rightarrow e^y = \frac{x^3}{3} - \frac{x^2}{2} + c \Rightarrow \underline{y = \ln(\frac{x^3}{3} - \frac{x^2}{2} + c)}$

since $y(0)=1$, then

$1 = \ln(\frac{0^3}{3} - \frac{0^2}{2} + c) \Rightarrow 1 = \ln(c) \Rightarrow \underline{c=e}$

So the solution is $y = \ln(\frac{x^3}{3} - \frac{x^2}{2} + e)$.

Continued on Page 8...

12. [6] Suppose that the bowl of candy in C-105 initially contains 100 pieces and let $y = y(t)$ stand for the number of pieces of candy in the bowl after t hours.

(a) Find an exact expression for $y(t)$ assuming that one-third of the pieces in the bowl are removed each hour, and so y satisfies the differential equation $\frac{dy}{dt} = -\frac{y}{3}$.

The function y will decrease according to the law of natural decay & so will have the form $y(t) = Ae^{-\frac{t}{3}}$,
Since $y(0) = 100$ & $y(0) = Ae^0 = A$, then $A = 100$,
so $y(t) = 100e^{-\frac{t}{3}}$.

(b) Now assume that Shelley is also continuously re-supplying the bowl at a rate of $\frac{75}{y}$ pieces per hour. Write a new differential equation that y satisfies in this case.

Since y is increased by $\frac{75}{y}$ pieces per hour, (continuously) then y satisfies the d.e. $\frac{dy}{dt} = -\frac{y}{3} + \frac{75}{y}$

(c) Find the equilibrium amount of candy in the bowl in this situation.

Solve $\frac{dy}{dt} = 0$: $-\frac{y}{3} + \frac{75}{y} = 0$

$\Rightarrow \frac{75}{y} = \frac{y}{3}$

$\Rightarrow 3 \cdot 75 = y^2$

$\Rightarrow y = \pm 15$

$\Rightarrow y = 15$ since in this situation $y = -15$ is not allowed.

Continued on Page 9...

13. Compute the following limits, or show that they do not exist. Justify your answers.

(a) [3] $\lim_{n\to\infty} \frac{e^n}{n!}$

$\frac{e^n}{n!} = \frac{e\cdot e\cdot e\cdot e \cdots e\cdot e}{1\cdot 2\cdot 3\cdot 4 \cdots (n-1)(n)}$

$< \left(\frac{e}{1}\right)\left(\frac{e}{2}\right)\left(\frac{e}{n}\right)$ since for $k \geq 3$, $\frac{e}{k} < 1$

$= \left(\frac{e^3}{2}\right)\cdot\frac{1}{n}$

So $0 \leq \frac{e^n}{n!} \leq \frac{e^3}{2}\left(\frac{1}{n}\right)$. Since $\lim_{n\to\infty} \frac{e^3}{2}\left(\frac{1}{n}\right) = 0$, then by the Squeeze Theorem, $\lim_{n\to\infty} \frac{e^n}{n!} = 0$.

(b) [3] $\lim_{n\to\infty} \frac{\ln(3n)}{\ln(n)} = \lim_{x\to\infty} \frac{\ln(3x)}{\ln(x)} \overset{L.H.}{=} \lim_{x\to\infty} \frac{\frac{1}{3x}(3)}{\frac{1}{2x}(2)} = 1$

So $\lim_{n\to\infty} \frac{\ln(3n)}{\ln(n)} = 1$

indeterminate of type $\infty - \infty$

(c) [4] $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right)$.

$= \lim_{x\to 0} \frac{\sin(x) - x}{x\sin(x)}$ ← indeterminate of type $\frac{0}{0}$

$\overset{L.H.}{=} \lim_{x\to 0} \frac{\cos(x) - 1}{\sin(x) + x\cos(x)}$

$\overset{L.H.}{=} \lim_{x\to 0} \frac{-\sin(x)}{\cos(x) + \cos(x) - x\sin(x)}$

$= \frac{0}{1+1-0} = 0$

So $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right) = 0$

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14.(a) [2] Define the term "absolute convergence" for a series $\sum_{n=1}^{\infty} a_n$.

The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

(b) [3] Give an example of a series that is convergent, but not absolutely convergent.

The series $\sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{1}{n}\right)$ is convergent. ~~and~~ This can be shown by using the Alternating Series Test.

The series $\sum_{n=1}^{\infty} \left|(-1)^{n-1}\left(\frac{1}{n}\right)\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is the Harmonic Series & so is divergent. Thus $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ is convergent, but not absolutely convergent.

(c) [3] Determine if the series $\sum_{n=1}^{\infty} \frac{(-7)^n}{n6^n}$ is absolutely convergent.

Use the Ratio Test:

$a_n = \frac{(-7)^n}{n6^n}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-7)^{n+1}}{(n+1)6^{n+1}}}{\frac{(-7)^n}{n6^n}}\right|$

$= \lim_{n\to\infty} \left|\frac{(-7)^{n+1}}{(-7)^n}\cdot\frac{6^n}{6^{n+1}}\cdot\frac{n}{n+1}\right| = \lim_{n\to\infty} \frac{7}{6}\left(\frac{n}{n+1}\right) = \frac{7}{6}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)$

$= \frac{7}{6}.$

Since $\frac{7}{8} > 1$, then by the Ratio Test this series is divergent & so is not absolutely convergent.

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15. Consider the power series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2 2^n}$.

(a) [4] Determine the radius of convergence of the power series.

Compute $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-5)^{n+1}}{(n+1)^2 2^{n+1}}}{\frac{(x-5)^n}{n^2 2^n}}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(x-5)^n}\frac{2^n}{2^{n+1}}\frac{n^2}{(n+1)^2}\right|$

$= \lim_{n\to\infty}\frac{|x-5|}{2}\frac{n^2}{(n+1)^2} = \frac{|x-5|}{2}$. $\frac{|x-5|}{2} < 1$ when $|x-5| < 2$

So the Radius of convergence is $R=2$

(b) [4] Determine the interval of convergence of the power series.

Check for convergence at $x = 5+2 = 7$ & $x = 5-2 = 3$:

$x=7$: $\sum_{n=1}^{\infty}\frac{(7-5)^n}{n^2 2^n} = \sum_{n=1}^{\infty}\frac{2^n}{n^2 2^n} = \sum_{n=1}^{\infty}\frac{1}{n^2}$ ← a convergent p-series.

$x=3$: $\sum_{n=1}^{\infty}\frac{(3-5)^n}{n^2 2^n} = \sum_{n=1}^{\infty}\frac{(-2)^n}{n^2 2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ ← a convergent alternating series.

So, the interval of convergence is $[3,7]$.

16. [5] Find the first four terms of the Maclaurin series for $f(x) = \frac{1}{\sqrt{1+2x}} = (1+2x)^{-1/2}$

⇒ The Maclaurin series is $f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$f(0) = \frac{1}{\sqrt{1+0}} = 1$, $f'(x) = -\frac{1}{2}(1+2x)^{-3/2}(2)$, so $f'(0) = -1$
$= -(1+2x)^{-3/2}$

$f''(x) = (-\frac{3}{2})(-1)(1+2x)^{-5/2}(2)$ ⇒ $f''(0) = 3$
$= 3(1+2x)^{-5/2}$

$f'''(x) = 3(-\frac{5}{2})(1+2x)^{-7/2}(2)$ so $f'''(0) = -15$
$= -15(1+2x)^{-7/2}$

Thus the Maclaurin series for $f(x)$ starts as:

$1 - x + \frac{3}{2}x^2 - \frac{15}{6}x^3 + \cdots$
$= 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \cdots$

* The Binomial Theorem could also be used

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17. (a) [3] State the Maclaurin series of the function $f(x) = \cos x$. You do not need to derive the series.

The Maclaurin series for cos(x) is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots,$$ (with radius of convergence $R=\infty$)

(b) [3]Find the Maclaurin series for the function $g(x) = \dfrac{1-\cos(x)}{x^2}$. Hint: Use your answer from (a).

The Maclaurin Series for g(x) is $\dfrac{\left(1 - \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right)}{x^2}$

$$= \frac{1}{x^2}\left(1 - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots + \frac{(-1)^{n-1} x^{2n}}{(2n)!} + \cdots\right)$$

$$= \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \cdots + \frac{(-1)^{n-1} x^{2n-2}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+2)!}$$

(c) [2] Use your answer from (b) to express $\int_0^1 \frac{1-\cos(x)}{x^2}\,dx$ as the sum of a series.

$$\int_0^1 \frac{1-\cos(x)}{x^2}dx = \int_0^1 \left(\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \cdots\right)dx$$

$$= \left(\frac{x}{2!} - \frac{x^3}{3\cdot 4!} + \frac{x^5}{5\cdot 6!} - \frac{x^7}{7\cdot 8!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+2)!} + \cdots\right)\Big|_0^1$$

$$= \frac{1}{2!} - \frac{1}{3\cdot 4!} + \frac{1}{5\cdot 6!} - \frac{1}{7\cdot 8!} + \cdots + \frac{(-1)^n}{(2n+1)(2n+2)!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)!}$$

(d) [2] The sum of the first two terms of the series from (c) provides an approximation of the definite integral from (c). Give an estimate for the error of this approximation.

Since the series from (c) satisfies the conditions of the Alternating Series Test then the third term in the series, $\frac{1}{5\cdot 6!}$ provides an estimate for the error given by $S_2 = \frac{1}{2!} - \frac{1}{3\cdot 4!}$, the sum of the first two terms.

$$\frac{1}{5\cdot 6!} = \frac{1}{5\cdot 720} = \frac{1}{3600} = 0.0002777\cdots$$

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