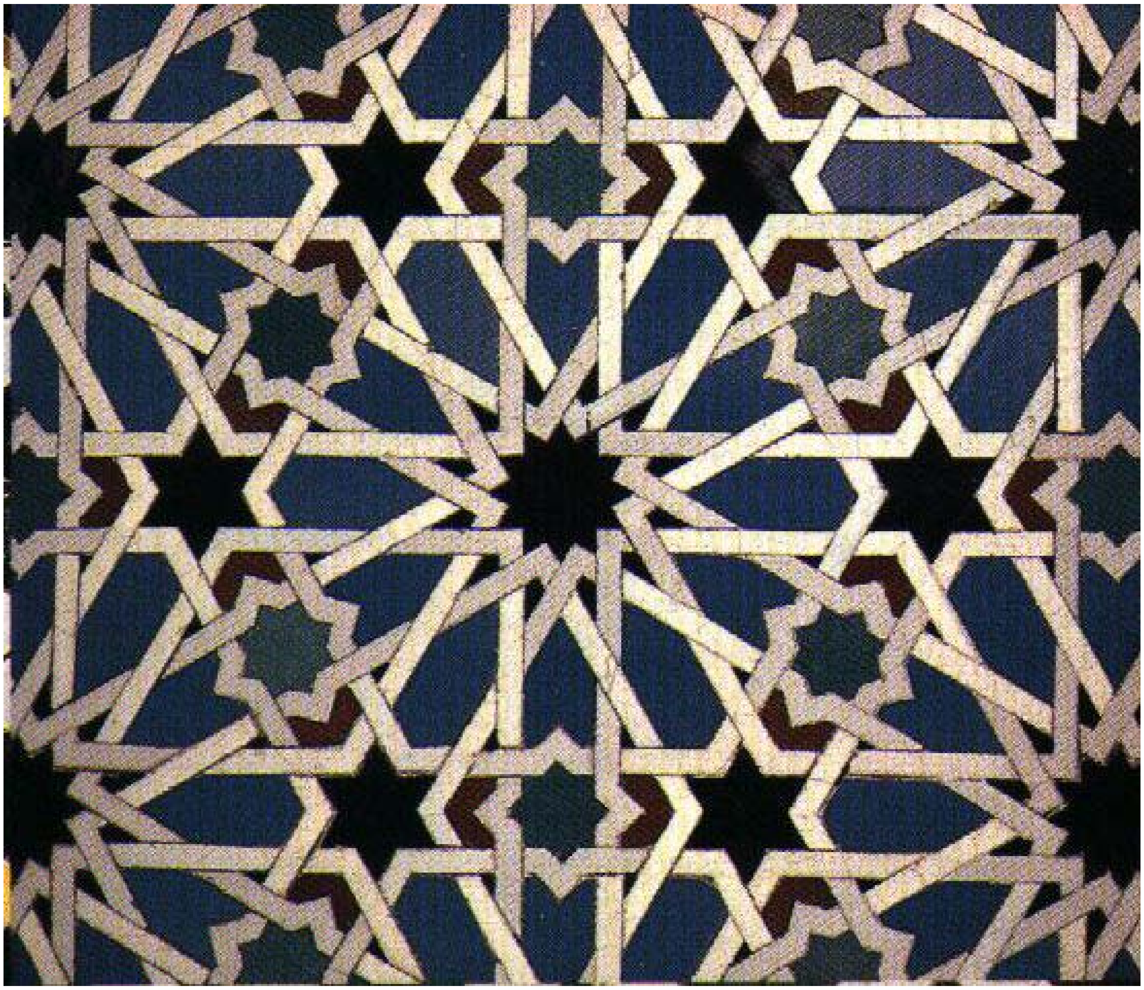
McMaster University

Arts and Science 1D6



Drs Deirdre Haskell & Matt Valeriote
Winter 2013

http://www.math.mcmaster.ca/~haskell/a&s1d_12-13/a&s1d-webpage.html

Front cover:

Tile pattern from Alhambra Palace, Granada, Spain

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Disclaimer

Information contained in this Course Package is subject to change. Changes and corrections will be announced in class and on the course web page

http://www.math.mcmaster.ca/~haskell/a&s1d_12-13/a&s1d-webpage.html

This course package was prepared in December 2012 and does not reflect any changes made after that date.

Important Dates

Important Dates

Week 1 * January 7-11

Monday, January 7: Classes begin. Work on **assignment 10**

Week 2 * January 14-18

Work on **assignment 11**
Quiz week
Tuesday, January 15: Last day for registration and adding or dropping courses.

Week 3 * January 21-25

Work on **assignments 11 and 12**

Week 4 * January 28-February 1

Quiz week
Work on **assignments 12 and 13**

Week 5 * February 4-8

Work on **assignments 13 and 14**

Week 6 * February 11-16

Quiz week
Work on **assignment 14**

February 18-22

Reading week, no classes.

Week 7 * February 25-March 1

Work on **assignment 15**

Week 8 * March 4-8

Tuesday, March 5: Test 2. Details (material covered, times and locations) will be announced on the course web page.

Work on **assignment 15**

Week 9 * March 11-15

Work on **assignment 16**

Friday, March 15: Last day for canceling courses without failure by default.

Week 10 * March 18-22

Quiz week
Work on **assignments 16 and 17**

Week 11 * March 25-29

Work on **assignments – from now on, it will be advertised in class and on the web page**

Friday, March 29: Good Friday, no classes.

Week 12 * April 1-5

Quiz week
Work on assignments

Week 13 * April 8-10

Wednesday, April 10: Classes end.

Detailed final exam information will be posted on the course web page.

Final exams: April 12-30, 2013

Deferred exams: June 17-21, 2013

Suggested Practice Questions

- lectures might not follow the order as listed below
- the section and question numbers refer to the 4th edition of the text. For those using the 3rd edition, many of the question numbers are the same. Below, a number (or number range) that appears within square brackets should be used instead of the preceding number (or number range) if you are using the 3rd edition of the text.

DIFFERENTIAL EQUATIONS

Section 7.1	1-11 odd, 15 [13]
Section 7.2	1-9 odd
Section 7.3	1-15 odd
Section 7.4	3, 9, 11, 13, 17, 19
Section 7.5	1, 3, 5, 7, 9
Section 7.6	1, 5, 7 [1, 3, 5]

SEQUENCES AND SERIES

Section 8.1	3-13 odd, 14-18 all, 25, 27 [21, 23]
Section 8.2	1, 9, 10, 11-37 odd, [11-31 odd], 58-63 all [48-53 all]
Section 8.3	1, 3, 5, 6-10 all, 11-23 odd
Section 8.4	2, 3-11 odd, [3-9 odd], 15, 21-33 odd, 37a, 37c [13, 19-29 odd, 33a, 33c]
Section 8.5	1-17 odd, 25 [1-19 odd]

Section 8.6	3-7 all, 9-15 odd, 23, 25, 27 [21, 23, 25]
Section 8.7	5-15 odd [5-13 odd], 25-33 odd [19-25 odd], 39, 43-53 odd, 59, 63 [31-43 odd, 49, 51]
Section 8.8 [8.9]	3, 5, 7, 11, 13, and in the 4th edition, 21-24
[Section 8.8	1, 3, 5, 6]

FUNCTIONS OF SEVERAL VARIABLES

Section 11.1	1-9 odd, 12 [10], 15-25 odd [13-21 odd], 39, 40 [35, 36]
Section 11.2	5, 6, 7-13 odd, 27, 29, 31, 35 [25, 27, 29, 33]
Section 11.3	1, 3, 6, 9, 15, 17, 23, 27, 31, 41, 44, 47, 55, 57, 70b, 70c [1, 3, 6, 7, 13, 15, 19, 23, 27, 37, 40, 43, 51, 53, 64b, 64c]
Section 11.4	1, 3, 11, 13 [9, 11]

N.B. You should also do questions 7 and 8 from assignment #7.

Name: ____________________

Arts Sci 1D06 **Winter 2012** **Assignment 13**

Material covered: Sections 7.1, 7.2, 7.3, 7.4.

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1. Show that the function $y = \frac{1}{2}(e^x + e^{-x})$ satisfies the differential equation $y'' = \sqrt{1 + (y')^2}$. Does $y = \frac{1}{2}(e^x - e^{-x})$ satisfy the same equation?

Continued on next page

2. Describe the following events as initial value problems (i.e., in each case write down a differential equation and an initial condition). Do not solve the equations.
(a) Ice starts forming at time $t = 0$. Let $T(t)$ be the thickness of the ice at time t. The rate at which ice is formed is inversely proportional to the square of its thickness.

(b) At time $t = 0$ somebody starts spreading the rumour that McMaster campus has been attacked by the Borg. Assume that there are $10,000$ students on the campus, and denote by $S(t)$ the number of people who have heard the rumour at time t. The rate of increase in the number of people who have heard the rumour is proportional to the number of people who have heard it and to the number of people who haven't heard it yet.

(c) A pie, initially at the temperature of 20^oC, is put into an 300^oC oven. Let $T(t)$ be the temperature of the pie at time t. The temperature of the pie changes proportionally to the difference between the temperature of the oven and the temperature of the pie.

Continued on next page

3. Solve the initial value problem $y' = \dfrac{1}{x^2y - 2x^2 + y - 2}, \qquad y(0) = 1.$

Continued on next page

4. Consider the initial value problem $y' = (y^2 + 1)x, \quad y(0) = 1$. Answer questions (a) and (b) WITHOUT SOLVING THE EQUATION:

(a) Find the intervals where the solution y is increasing and decreasing.

(b) Find all relative extreme values of y.

(c) Solve the given equation algebraically.

Continued on next page

5. The equation $2xyy' = y^2 - x^2$ is not separable. Show that, by introducing the new function $v = y/x$, the above equation can be reduced to a separable equation. Solve that equation, thus solving the original equation.

6. Repeat the previous exercise for the equation $xy' \sin(y/x) = y \sin(y/x) - x$.

Continued on next page

7. A population is modeled by the differential equation

$$\frac{dP(t)}{dt} = 1.42P(t)\left(1 - \frac{P(t)}{5600}\right).$$

(a) For what values of $P(t)$ is the population increasing? Decreasing?

(b) Explain what is an equilibrium solution. What are the equilibrium solutions of the given equation?

(c) Sketch the solutions of the given equation with initial conditions $P(0) = 2000$ and $P(0) = 10000$.

THE END

Name: ____________________

Arts Sci 1D06 Winter 2012 Assignment 14

Material covered: Sections 7.4, 7.5, 7.6.

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1. Solve the initial value problem $y' - 2xy = x$, $y(0) = 2$.

2. Solve the initial value problem $y' \cos\theta - y \sin\theta = \cos\theta$, $-\pi/2 \leq \theta \leq \pi/2$, $y(0) = 1$.

Continued on next page

3. The change in populations of red-footed foxes and white-tailed brown rabbits can be described by the following set of equations: $\frac{dx}{dt} = 0.2x - 0.001xy$, $\frac{dy}{dt} = -0.4y + 0.0000016xy$.
(a) Which of the populations, foxes or rabbits, is described by x? Which one is described by y? Explain your answer.

(b) Find equilibrium solutions and explain their meaning.

(c) Find an expression for dy/dx and interpret it as a differential equation.

Continued on next page

4. The half-life of a radioactive substance is 12 years. Suppose we have a 1000 grams sample.

(a) Find the mass that remains after t years.

(b) Estimate the time needed for the substance to decay to 100 grams.

(c) Find the time needed for the substance to decay to 15 % of its original amount.

Continued on next page

5. [Context of question 11 on page 539] The half life of the carbon ^{14}C is 5730 years. An object was found, that contains 16 % as much ^{14}C radioactivity as the corresponding material on Earth today.
(a) Estimate the age of the object.

(b) Assume that the measurement of the ^{14}C radioactivity was off by 1 % (i.e., the object contains 15-17 % as much ^{14}C radioactivity as the corresponding material on Earth today). Give an estimate (in terms of an interval) of the age of the object.

Continued on next page

6. Solve the equation

$$\frac{dP}{dt} = 3P\left(1 - \frac{P}{10}\right), \qquad P(0) = 1.$$

Continued on next page

7. The population of long-nosed-short-eared amber-brown ants has been modeled by the differential equation

$$\frac{dA}{dt} = kA\left(1 - \frac{A}{K}\right),$$

where $A(t)$ is the biomass in kilograms at time t (biomass is the total mass of all members of population). The carrying capacity is estimated to be K=60000 kilograms, and k=0.66 per year.

(a) If $A(0) = 10000$, find the biomass two years later.

(b) If $A(0) = 10000$, find the biomass ten years later.

(c) How long will it take for the biomass to reach 59000 kilograms?

THE END

Name: ______________________________

Arts Sci 1D06　　Winter 2012　　Assignment 15

Material covered: Sections 8.1, 8.2.

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1. (a) Find the limit of the sequence $a_n = 3\ln(4n+3) - \ln(n^3-1)$.

(b) Determine whether the sequences $b_n = \sin(n\pi/2)$ and $c_n = \cos(n\pi/2)$ are convergent or not. If they are convergent, find their limit.

Continued on next page

2. (a) True or false: if the sequence $\{a_n\}$ is convergent and the sequence $\{b_n\}$ is divergent then the sequence $\{a_n b_n\}$ is convergent.

(b) True or false: if $\lim_{n\to\infty} a_n = 0$, then $\sum_{i=1}^{\infty} a_n$ is convergent.

3. Find the limit of the sequence $a_n = \dfrac{4^n n!}{(n+2)!}$.

Continued on next page

4. It is known that $\lim_{n\to\infty}(0.6)^n = 0$ (why?). Find n such that $(0.6)^n < 10^{-10}$.

5. Determine whether the series $\displaystyle\sum_{n=0}^{\infty}\frac{6^{2n+1}}{3^{4n+1}}$ is convergent or not. If it is convergent, find its sum.

Continued on next page

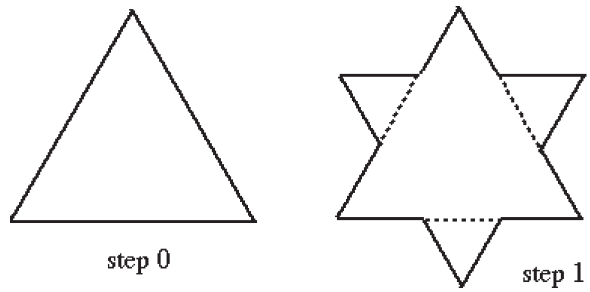
6. (a) Is the series $\sum_{n=10}^{\infty} \arctan\left(\frac{n^3}{n^3 - n}\right)$ convergent or not?

(b) Determine whether the series $\sum_{n=0}^{\infty} \frac{\sqrt{n^3+1}}{(n^2+13)^2}$ is convergent or not.

(c) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{e^{3n}}$.

Continued on next page

7. Start with an equilateral triangle of side 1 (call that step 0). Divide each side into three equal parts and construct an equilateral triangle over the middle part (that is step 1). In step 2, repeat the above process by building an equilateral triangle over each of the 12 sides. If you keep repeating this process indefinitely, you will obtain the Koch snowflake curve.



(a) Find the number of sides of the polygon that is obtained in the n-th step. Find the length of each side and the total length of the curve.

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(b) Find the length of the snowflake curve.

(c) Find the area of the region bounded by the snowflake curve.

THE END

Name: ____________________

Arts Sci 1D06 Winter 2012 Assignment 16

Material covered: Sections 8.3, 8.4, 8.5, 8.6.

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1. Consider the series $\sum_{n=1}^{\infty} ne^{-2n}$.

(a) Check that all asumptions of the integral test are satisfied.

(b) Determine whether the given series is convergent or not.

Continued on next page

2. Determine whether the following series are convergent or not.

(a) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{3+e^n}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2-n-1}{3n^3+66n+1}$.

Continued on next page

3. Use the ratio test to determine whenter the following series converge or not.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{n\,3.99^n}$.

Continued on next page

4. True/false questions.

(a) If $\sum_{n=0}^{\infty} a_n 3^n$ is convergent, then $\sum_{n=0}^{\infty} a_n 4^n$ is convergent.

(b) If $\sum_{n=0}^{\infty} a_n 3^n$ is divergent, then $\sum_{n=0}^{\infty} a_n 4^n$ is divergent.

(c) If $\sum_{n=0}^{\infty} a_n 3^n$ is convergent, then $\sum_{n=0}^{\infty} a_n (-3)^n$ is convergent.

Continued on next page

5. (a) Find a power series representation of the function $f(x) = \dfrac{1}{3+4x}$.

(b) What is the radius of convergence of the series in (a)?

(c) Use (a) to find a power series representation of $\ln(3+4x)$.

(d) What is the radius of convergence of the series in (c)?

Continued on next page

6. Determine the radius of convergence for the following series

(a) $\sum_{n=0}^{\infty} \frac{x^n}{n\,14^n}$.

(b) $\sum_{n=0}^{\infty} n\,x^n$.

(c) $\sum_{n=0}^{\infty} n!\,x^n$.

Continued on next page

7. Determine the radius of convergence and the interval of convergence for the series

$$\sum_{n=0}^{\infty} \frac{3n}{5^n}(3x-1)^n.$$

Continued on next page

8. Use a power series to determine the value of the integral $\int_0^{0.1} \frac{1}{1+x^4}\, dx$ to four decimal places.

9. Evaluate $\int \arctan(x^3)\, dx$ as a power series.

THE END

Name: ______________________

Arts Sci 1D06 Winter 2012 Assignment 17

Material covered: Sections 8.6, 8.7, 8.8; few review questions about series.

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1. Find the Taylor series for the following functions.
(a) $f(x) = e^x$ centred at $x = 0$.

(b) $f(x) = e^x$ centred at $x = 1$.

(c) $f(x) = e^x$ centred at $x = -1$.

Continued on next page

2. (a) Write down the Maclaurin series expansion of e^x.

(b) Using (a), find the Maclaurin series expansion of xe^{x^2-2}.

(c) Write down the Maclaurin series expansions for $\sin x$ and $\cos x$.

(d) Using (c), find the Maclaurin series expansion of $\sin(x+1)$.

Continued on next page

3. (a) Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{2^{2n+1}(2n+1)!}$.

(b) Using series, find the limit $\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3 - \frac{1}{120}x^5}{x^7}$.

(c) Expand $(1 - x^3)^{1/3}$ as a power series and find its radius of convergence.

Continued on next page

4. (a) Evaluate the indefinite integral $\int \sin(x^3)\, dx$ as an infinite series.

(b) Evaluate the indefinite integral $\int e^{-x^2}\, dx$ as an infinite series.

Continued on next page

5. Express 5.4114114114114... as a fraction.

6. Is the series $\sum_{n=1}^{\infty} \frac{(n+4)!}{n!7^n}$ absolutely convergent?

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7. (a) Find an example of a series that is convergent but is not absolutely convergent.

(b) Is it true that if $\sum_{n=1}^{\infty} a_n$ is divergent, then the series $\sum_{n=1}^{\infty} |a_n|$ is also divergent?

8. How many terms of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n}$ do we have to add in order to find the sum up to an error of less than 0.001?

THE END

Name: ______________________

Arts Sci 1D06	**Winter 2012**	**Assignment 18**

Material covered: Sections 11.1, 11.2, 11.3.

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1. (a) Sketch the domain of the function $f(x, y) = (x^2 - y^2)^{-1/2}$.

(b) Sketch the domain and find the range of the function $f(x, y) = \arcsin(xy)$.

Continued on next page

2. Draw a contour map of the given function, showing (and labeling) several level curves.

(a) $f(x, y) = xy$

Continued on next page

(b) $f(x, y) = e^{1/(x^2+y^2)}$.

Continued on next page

3. Find the range of the function $f(x, y, z, t) = \dfrac{xy - ze^t}{x^2 + y^2}$.

4. Find the domain and sketch the graph of the function $f(x, y) = \sqrt{4 - x^2 - 2y^2}$.

Continued on next page

5. (a) Determine the largest set on which the function $z = \ln x \ln y$ is continuous.

(b) Determine the largest set on which the function $z = \ln(3x - y)$ is continuous.

(c) Find all points where the function $f(x, y) = \cos(x - y) + \sec(x - y)$ is not continuous.

Continued on next page

6. Let $f(x,y) = \int_{xy}^{2} e^t dt + \int_{x}^{x^2} t^2 dt.$

(a) Find $f(0,1)$.

(b) Write down the statement of the Fundamental Theorem of Calculus, Part I.

(c) Find $f_x(x,y)$ and $f_y(x,y)$.

Continued on next page

7. (a) Does the function $u(x,t) = \sin(x-2t) + 3\ln(x+2t)$ satisfy the wave equation $u_{tt} = 4u_{xx}$?

(b) The kinetic energy of a body of mass m and velocity v is $K = \frac{1}{2}mv^2$. Show that $\dfrac{\partial K}{\partial m}\dfrac{\partial^2 K}{\partial v^2} = K$.

(c) Use differentials to approximate $f(0.03, 2.94)$, where $f(x,y) = \sqrt{1-x^2+y}$.

Continued on next page

8. Let N_{Mac} be the number of people who consider buying a Macintosh computer and let N_{PC} be the number of people who consider buying a comparable PC. P_{Mac} and P_{PC} are the prices of a Macintosh and a PC respectively. Find the signs of

$$\frac{\partial N_{Mac}}{\partial P_{PC}} \qquad \text{and} \qquad \frac{\partial N_{PC}}{\partial P_{Mac}}.$$

THE END

Arts & Science 1D6 Test #2

Day Class
Test #2
Duration of test: 60 minutes
McMaster University
1 March, 2011

Dr. Matt Valeriote

Last Name:______________________________ Initials:__________

Student No.:______________________________ Your TA's Name:______________

This test has 8 pages and 7 questions and is printed on BOTH sides of the paper. Pages 7 and 8 contain no questions and can be used for rough work.

You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancies to the attention of the invigilator.

Attempt all questions and write your answers in the space provided.

Marks are indicated next to each question; the total number of marks is 40.

Any Casio fx991 calculator is allowed. Other aids are not permitted.

Use pen to write your test. If you use a pencil, your test will not be accepted for regrading (if needed).

Good Luck!

Score

Question	1	2	3	4	5	6	7	Total
Points	6	5	6	6	6	6	5	40
Score								

continued . . .

ALL QUESTIONS: you must show your work to receive full credit.

1. [6] Let $g(x) = \int_0^{x^2} te^{-t}\, dt$.

   (a) Compute $g'(x)$.

   (b) Find the interval(s) on which the function $g(x)$ is concave upward

2. [5] Determine if the following improper integral is convergent and evaluate it if it is.

$$\int_0^2 \frac{x}{4 - x^2}\, dx.$$

continued . . .

3. [6] Evaluate the following indefinite integrals.

   (a) $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}\,dx$

   (b) $\displaystyle\int 4x\cos(4x)\,dx$

4. [6] Let R be the region bounded by the curve $y = x(x-1)^2$ and the x-axis.

   (a) Find the area of R.

continued ...

(b) Set up but DO NOT EVALUATE the integral representing the volume obtained by rotating the region R about the line $y = -1$.

5. [6] Solve the initial value problem $(x+1)\dfrac{dy}{dx} = y^2$, $y(0) = \frac{1}{2}$ for $x > -1$.

continued . . .

6. [6] A cup of coffee is poured from a pot whose contents are 95°C into a non-insulated cup in a room at 20°C. Let $T(t)$ be the temperature of the coffee after t minutes. Assuming that the coffee cools according to Newton's Law, then

$$\frac{dT}{dt} = k(T - 20).$$

(a) Solve this differential equation subject to the initial condition $T(0) = 95$.

(b) After one minute, the coffee has cooled to 90°C. Use this information and your solution to (a) to solve for the constant k.

continued ...

7. [6] Populations of aphids (A) and ladybugs (L) are modeled by the predator-prey equations

$$\begin{aligned}\frac{dA}{dt} &= 2A - 0.01AL \\ \frac{dL}{dt} &= -0.5L + 0.0001AL\end{aligned}$$

(a) Find the equilibrium solutions and explain their significance.

(b) When there are 1000 aphids and 300 ladybugs, is the aphid population increasing or decreasing? Justify your answer.

continued . . .

Name ______________________________

Student Number ______________________________

Your TA's Name: ______________________________

Arts & Science 1D06

DAY CLASS
APRIL EXAM
DURATION OF EXAM: 3 Hours
MCMASTER UNIVERSITY

DR. MATT VALERIOTE

12 April, 2011

THIS EXAMINATION PAPER INCLUDES 14 PAGES AND 17 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF YOUR INVIGILATOR.
Attempt all questions.
The total number of available points is 100.
Marks are indicated next to each question.
Use of a Casio fx991 calculator only is allowed.
Write your answers in the space provided.
You must show your work to get full credit.
Use the last two pages for rough work.
Good Luck.

Score

Question	1–3	4–6	7	8	9	10	11
Points	9	9	8	6	9	7	7
Score							

Question	12	13	14	15	16	17	Total
Points	8	6	4	10	9	8	100
Score							

Continued on Page 2 . . .

Multiple Choice Questions

Indicate your answers to questions 1–3 by circling only ONE of the letters. Each of these questions is worth 3 marks.

1. [3] $\lim_{x\to 0^+} x^{x^2}$ is equal to

(A) 0 (B) 1 (C) e (D) does not exist

2. [3] Which of the following three tests can be used to show that the series $\sum_{n=1}^{\infty} \frac{3}{n(n+2)}$ is convergent?

(I) The Ratio Test.

(II) The Comparison Test with $\sum_{n=1}^{\infty} 3n^{-2}$.

(III) The Limit Comparison Test with $\sum_{n=1}^{\infty} 3n^{-1}$.

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three

Continued on Page 3. . .

3. [3] Let $f(x) = \int_0^{\cos x} \sqrt[3]{1-t^2}\, dt$. Then $f'(x)$ is:

(A) $\dfrac{1}{3(1-\cos^2 x)^{2/3}}$ (B) $-\sin x \sqrt[3]{1-\cos^2 x}$

(C) $\sqrt[3]{1-\cos^2 x}$ (D) $\dfrac{-2\sin x \cos x}{3(1-\cos^2 x)^{2/3}}$

Continued on Page 4. . .

True/False Questions.

Decide whether the statements in questions 4–6 are true or false by circling your choice. YOU MUST JUSTIFY YOUR ANSWER TO RECEIVE FULL CREDIT. Each of these questions is worth 3 marks.

4. [3] If $f'(x)$ exists and is nonzero for all x then $f(1) \neq f(0)$.

TRUE FALSE

5. [3] The differential equation $y' = x^2 + y^2 + 1$ has an equilibrium solution.

TRUE FALSE

6. [3] If b_n is a sequence of positive numbers such that $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges.

TRUE FALSE

Continued on Page 5 . . .

Questions 7–17: you must show work to receive full credit

7. Consider the function $f(x) = x\sqrt{1-x^2}$.

(a) [2] Find the x-intercepts of $f(x)$.

(b) [3] Compute $f'(x)$.

(c) [3] On which interval(s) is $f(x)$ increasing?

8. [6] Find the area of the region enclosed by the curves $y = 4 - x^2$ and $y = x^2 + 2$.

Continued on Page 6...

9. Compute the following integrals.

   (a) [2] $\displaystyle\int_3^4 \frac{x}{\sqrt{25-x^2}}\,dx$.

   (b) [3] $\displaystyle\int \ln(1+x^2)\,dx$. Hint: Use Integration by Parts.

   (c) [4] $\displaystyle\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$.

Continued on Page 7…

10. The growth of a population of mice is given by the differential equation

$$\frac{dP}{dt} = (0.5)P\left(1 - \frac{P}{500}\right),$$

with time measured in months. Assume that the initial size of the population is 5.

(a) [3] Write a formula for the population $P(t)$. You may use that the general solution of the logistic equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$ is $\dfrac{K}{1 + Ae^{-kt}}$ for some constant A.

(b) [4] How many months does it take for the population to climb to 100?

11.[7] Find the solution of the differential equation $xy\dfrac{dy}{dx} = x + 1$, for $x > 0$, that satisfies the initial condition $y(1) = 2$.

Continued on Page 8. . .

12. Consider the differential equation

$$xy' = 4x^3 - y.$$

(a) [2] By rewriting this differential equation, show that it is linear.

(b) [3] Find an integrating factor for this differential equation.

(c) [3] Find the solution of this differential equation subject to the condition $y(1) = 0$.

Continued on Page 9. . .

13. Determine whether the sequence converges or diverges. If it converges, find the limit.

(a) [3] $a_n = \dfrac{3n}{e^{(3/n)}}$

(b) [3] $a_n = \dfrac{n \sin n}{(n^2+1)}$

14. [4] Write out the first five terms of the Maclaurin series of the function $\dfrac{1}{(1+x)^3}$.

Continued on Page 10. . .

15. Consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{3n+4}$.

(a) [4] Show that the series is convergent.

(b) [3] If S is the sum of the series, provide an estimate for the difference between S and S_{99}, the 99th partial sum of the series. Do not calculate S_{99}.

(c) [3] Is the series absolutely convergent? Justify your answer to receive credit.

Continued on Page 11...

16. Consider the power series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n(n+1)}$.

(a) [6] Determine the radius of convergence of the power series.

(b) [3] Determine the interval of convergence of the power series.

Continued on Page 12. . .

17. (a) [3] Give the Maclaurin series of the function $f(x) = \cos x$.

(b) [3] Find the Maclaurin series for the function $g(x) = \cos(x^2)$. Hint: Use your solution from (a).

(c) [2] Evaluate $\int \cos(x^2)\,dx$ as an infinite series.

Continued on Page 13. . .

Arts & Science 1D06 Test #2

Day Class
Test #2
Duration of test: 60 minutes
McMaster University
28 February, 2012

Dr. Matt Valeriote

Last Name:______________________________ Initials:__________

Student No.:______________________________ Your TA's Name:______________

This test has 8 pages and 7 questions and is printed on BOTH sides of the paper. Pages 7 and 8 contain no questions and can be used for rough work.

You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancies to the attention of the invigilator.

Attempt all questions and write your answers in the space provided.

Marks are indicated next to each question; the total number of marks is 40.

Any Casio fx991 calculator is allowed. Other aids are not permitted.

Use pen to write your test. If you use a pencil, your test will not be accepted for regrading (if needed).

Good Luck!

Score

Question	1	2	3	4	5	6	7	Total
Points	7	6	5	4	6	6	6	40
Score								

continued ...

ALL QUESTIONS: you must show your work to receive full credit.

1.(a) [4] State both parts of the Fundamental Theorem of Calculus

(b) [3] Use the Fundamental Theorem of Calculus to find the derivative of the function

$$y = \int_{\sin x}^{\cos x} (1 + v^2)^{10}\, dv.$$

continued . . .

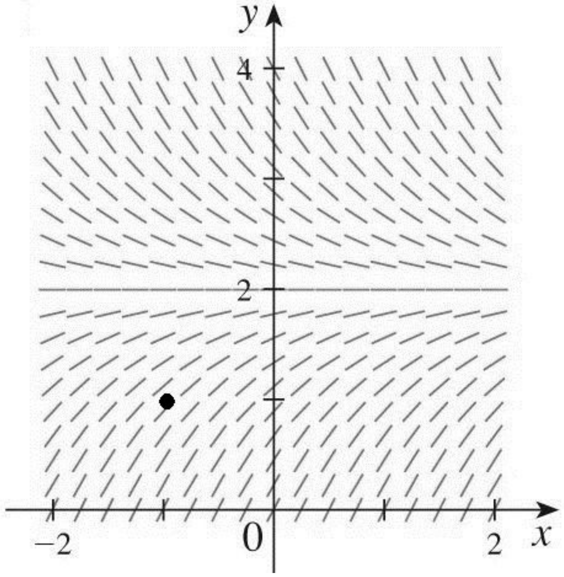
2. [6] Find $\int \sin\sqrt{x}\,dx$. [Hint: After making a suitable substitution, use integration by parts.]

3. [5] Let $p > 1$. Show that the following improper integral is convergent and evaluate it.

$$\int_1^\infty \frac{1}{x^p}\,dx.$$

continued . . .

4. [4] A direction field for a particular differential equation is given below.



(a) On the direction field, sketch the graph of the solution to the differential equation that passes through the point $(-1, 1)$.

(b) Which of the following differential equations matches the given direction field? Circle your answer.

(A) $\dfrac{dy}{dx} = 2 - y$ (B) $\dfrac{dy}{dx} = x^2 - y^2$ (C) $\dfrac{dy}{dx} = y - 1$ (D) $\dfrac{dy}{dx} = x^2 + 2$

continued . . .

5. [6] Let R be the region bounded by $x = 0$, $y = 1$, and the curve $y = \frac{x^3}{8}$. Sketch the region R and set up an integral for the volume of the solid obtained by rotating R about the horizontal line $y = -2$. **DO NOT EVALUATE THE INTEGRAL!**

6. [6] Solve the differential equation: $\frac{dy}{dx} = \frac{xy + 3x}{x^2 + 1}$.

continued . . .

7. [6] Let $P(t)$ be the population of the Earth t years after the year 2000 and assume that $P(t)$ grows according to the logistic equation

$$\frac{dP}{dt} = (0.02)P\left(1 - \frac{P}{60}\right).$$

In the year 2000 the population of the Earth was 6 billion people (so $P(0) = 6$).

(a) Write a formula for the population $P(t)$. You may use that the general solution of the logistic equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ is $\dfrac{M}{1 + Ae^{-kt}}$ for some constant A.

(b) What is the projected population of the Earth in the year 2100?

continued . . .

Name ______________________________

Student Number ______________________________

Your TA's Name: ______________________________

Arts & Science 1D06

DAY CLASS
APRIL EXAM
DURATION OF EXAM: 3 Hours
MCMASTER UNIVERSITY

DR. MATT VALERIOTE

10 April, 2012

THIS EXAMINATION PAPER INCLUDES 14 PAGES AND 17 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF YOUR INVIGILATOR.
Attempt all questions.
The total number of available points is 100.
Marks are indicated next to each question.
Use of a Casio fx991 calculator only is allowed.
Write your answers in the space provided.
You must show your work to get full credit.
Use the last two pages for rough work.
Good Luck.

Score

Question	1–3	4–6	7	8	9	10	11
Points	9	9	4	10	8	6	7
Score							

Question	12	13	14	15	16	17	Total
Points	6	10	8	8	5	10	100
Score							

Continued on Page 2 . . .

Multiple Choice Questions

Indicate your answers to questions 1–3 by circling only ONE of the letters. Each of these questions is worth 3 marks.

1. [3] Let

$$f(x) = \frac{1}{e^x + 1}.$$

Which of the following statements are **true**?

(I) The domain of $f(x)$ is $(-\infty, \infty)$.

(II) $f(x)$ is an odd function.

(III) $f(x)$ has an inverse.

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

2. [3] Which of the following series are convergent?

$$\text{(I)} \quad \sum_{n=1}^{\infty} (-1)^n \qquad \text{(II)} \quad \sum_{n=1}^{\infty} 2^n \qquad \text{(III)} \quad \sum_{n=1}^{\infty} \frac{1}{2+n^3}$$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

Continued on Page 3. . .

3. [3] Let $g(x) = \int_{x^2}^{1} \sin(\sqrt{t})\, dt$. Then $g'(\pi/2)$ is:

(A) 0 (B) $-\pi$

(C) $\sin(1) - 1$ (D) -1

True/False Questions.

Decide whether the statements in questions 4–6 are true or false by circling your choice. YOU MUST JUSTIFY YOUR ANSWER TO RECEIVE FULL CREDIT. Each of these questions is worth 3 marks.

4. [3] There is some value of x in the interval $(2, 3)$ such that $x^3 - 5x - 7 = 0$.

TRUE FALSE

Continued on Page 4. . .

5. [3] The improper integral $\int_0^\infty \cos(x)\,dx$ is convergent.

TRUE FALSE

6. [3] If the power series $\sum_{n=1}^{\infty} c_n x^n$ converges when $x = -8$ then the series $\sum_{n=1}^{\infty} c_n 7^n$ converges.

TRUE FALSE

Questions 7–17: you must show work to receive full credit

7. [4] Sketch the region bounded by the curves $y = x^2$ and $y = 2x + 3$ and set up an integral for its area. **Do not evaluate the integral!**

Continued on Page 5 . . .

8.(a) [5] Let $f(x) = \dfrac{x}{1+x^2} + 1$. Find the absolute maximum and absolute minimum values of $f(x)$ on the interval $[-3, 2]$.

(b) [5] Find the intervals where $f(x) = \sin(2x) - 4\sin(x)$, $0 \le x \le \pi$, is concave up and concave down and identify all points of inflection.

Continued on Page 6. . .

9. Compute the following integrals.

(a) [4] $\int_0^1 (x+1)e^{-x}\,dx.$

(b) [4] $\int \frac{1}{(x+2)(x+3)}\,dx.$

Continued on Page 7...

10. [6] Consider the predator-prey system $x' = 4x - xy$, $y' = -y + \frac{xy}{2}$.

   (a) Which of the variables, x or y, represents the predator? Explain why.

   (b) For each of the species represented by x and y, explain what happens if the other is not present.

   (c) Find all equilibrium solutions of this system.

11.[7] Find the solution of the differential equation $\frac{dy}{dx} = \frac{(x^2 - x)}{e^y}$, that satisfies the initial condition $y(0) = 1$.

Continued on Page 8...

12. [6] Suppose that the bowl of candy in C-105 initially contains 100 pieces and let $y = y(t)$ stand for the number of pieces of candy in the bowl after t hours.

(a) Find an exact expression for $y(t)$ assuming that one-third of the pieces in the bowl are removed each hour, and so y satisfies the differential equation $\frac{dy}{dt} = -\frac{y}{3}$.

(b) Now assume that Shelley is also continuously re-supplying the bowl at a rate of $\frac{75}{y}$ pieces per hour. Write a new differential equation that y satisfies in this case.

(c) Find the equilibrium amount of candy in the bowl in this situation.

Continued on Page 9. . .

13. Compute the following limits, or show that they do not exist. Justify your answers.

(a) [3] $\lim_{n\to\infty} \frac{e^n}{n!}$

(b) [3] $\lim_{n\to\infty} \frac{\ln(3n)}{\ln(n)}$

(c) [4] $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right)$.

Continued on Page 10...

14.(a) [2] Define the term "absolute convergence" for a series $\sum_{n=1}^{\infty} a_n$.

(b) [3] Give an example of a series that is convergent, but not absolutely convergent.

(c) [3] Determine if the series $\sum_{n=1}^{\infty} \frac{(-7)^n}{n6^n}$ is absolutely convergent.

Continued on Page 11...

15. Consider the power series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2 2^n}$.

    (a) [4] Determine the radius of convergence of the power series.

    (b) [4] Determine the interval of convergence of the power series.

16. [5] Find the first four terms of the Maclaurin series for $f(x) = \frac{1}{\sqrt{1+2x}}$.

Continued on Page 12. . .

17. (a) [3] State the Maclaurin series of the function $f(x) = \cos x$. You do not need to derive the series.

(b) [3]Find the Maclaurin series for the function $g(x) = \dfrac{1 - \cos(x)}{x^2}$. Hint: Use your answer from (a).

(c) [2] Use your answer from (b) to express $\displaystyle\int_0^1 \frac{1 - \cos(x)}{x^2}\,dx$ as the sum of a series.

(d) [2] The sum of the first two terms of the series from (c) provides an approximation of the definite integral from (c). Give an estimate for the error of this approximation.

Continued on Page 13. . .

Arts & Science 1D6 Test #2

Day Class
Test #2
Duration of test: 60 minutes
McMaster University
1 March, 2011

Dr. Matt Valeriote

Last Name: Solutions

Initials:_________

Student No.:____________________________

Your TA's Name:____________

This test has 8 pages and 7 questions and is printed on BOTH sides of the paper. Pages 7 and 8 contain no questions and can be used for rough work.

You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancies to the attention of the invigilator.

Attempt all questions and write your answers in the space provided.

Marks are indicated next to each question; the total number of marks is 40.

Any Casio fx991 calculator is allowed. Other aids are not permitted.

Use pen to write your test. If you use a pencil, your test will not be accepted for regrading (if needed).

Good Luck!

Score

Question	1	2	3	4	5	6	7	Total
Points	6	5	6	6	6	6	5	40
Score								

continued ...

ALL QUESTIONS: you must show your work to receive full credit.

1. [6] Let $g(x) = \int_0^{x^2} te^{-t}\,dt$.

   (a) Compute $g'(x)$.

Use the Chain Rule & the F.T.C.:

$$g'(x) = (x^2)e^{-(x^2)} \cdot (x^2)'$$
$$= 2x^3 e^{-x^2}$$

   (b) Find the interval(s) on which the function $g(x)$ is concave upward

Compute $g''(x)$:

$$g''(x) = 6x^2 e^{-x^2} + (2x^3)e^{-x^2}(-2x)$$
$$= 6x^2 e^{-x^2} - 4x^4 e^{-x^2}$$
$$= 2x^2 e^{-x^2}[3-2x^2]$$

$g''(x) > 0$ when $3-2x^2 > 0$ or $x^2 < \frac{3}{2} \Leftrightarrow -\sqrt{\frac{3}{2}} < x < \sqrt{\frac{3}{2}}$

So $g(x)$ is concave upward on the interval $\left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$.

2. [5] Determine if the following improper integral is convergent and evaluate it if it is.

$$\int_0^2 \frac{x}{4-x^2}\,dx.$$

Since $\frac{x}{4-x^2}$ has a vertical asymptote at $x=2$, then this integral is improper.

$\int_0^2 \frac{x}{4-x^2}dx = \lim_{t\to 2^-} \int_0^t \frac{x}{4-x^2}dx$. Let $u = 4-x^2$. Then $du = -2x\,dx$

$$= \lim_{t\to 2^-} \int_4^{4-t^2} -\frac{1}{2u}\,du = \lim_{t\to 2^-} -\frac{1}{2}\ln|u| \Big|_4^{4-t^2}$$

$$= \lim_{t\to 2^-} -\frac{1}{2}\left(\ln|4-t^2| - \ln|4|\right) = -\frac{1}{2}\lim_{t\to 2^-} \ln|4-t^2| + \frac{\ln 4}{2}$$

$= \infty$, since $\lim_{t\to 2^-} \ln|4-t^2| = -\infty$

continued . . .

So, this integral is <u>divergent</u>.

3. [6] Evaluate the following indefinite integrals.

(a) $\int \frac{e^{2x}}{1+e^{2x}}\,dx$ Use the substitution $u=1+e^{2x}$

$du=2e^{2x}dx$

$=\frac{1}{2}\int\frac{du}{u}=\frac{1}{2}\ln|u|+C$

$=\frac{1}{2}\ln|1+e^{2x}|+C=\frac{1}{2}\ln(1+e^{2x})+C$ (since $1+e^{2x}>0$)

(b) $\int 4x\cos(4x)\,dx$ Integrate by parts: $u=x$ $dv=4\cos(4x)dx$

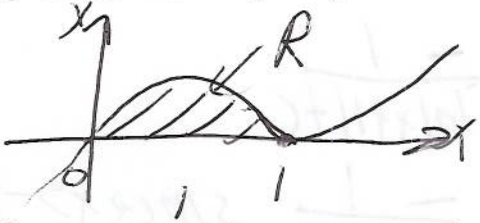
$du=dx$ $v=\sin(4x)$

$=(x)(\sin(4x))-\int\sin(4x)dx$

$=x\sin(4x)+\frac{1}{4}\cos(4x)+C$

4. [6] Let R be the region bounded by the curve $y = x(x-1)^2$ and the x-axis.

(a) Find the area of R.



Area of $R=\int_0^1 x(x-1)^2dx=\int_0^1(x^3-2x^2+x)dx$

$=\left(\frac{1}{4}x^4-\frac{2}{3}x^3+\frac{x^2}{2}\right)\Big|_0^1$

$=\frac{1}{4}-\frac{2}{3}+\frac{1}{2}=\frac{3}{12}-\frac{8}{12}+\frac{6}{12}$

$\boxed{=\frac{1}{12}}$

continued ...

(b) Set up but DO NOT EVALUATE the integral representing the volume obtained by rotating the region R about the line $y = -1$.

Volume

$= \int_0^1 \pi(1+x(x-1)^2)^2 dx - \int_0^1 \pi(1)^2 dx$

$= \pi \int_0^1 (x^3-2x^2+x+1)^2 dx - \int_0^1 \pi dx$

$= \pi \left[\int_0^1 (x^3-2x^2+x+1)^2 dx - 1 \right]$

5. [6] Solve the initial value problem $(x+1)\dfrac{dy}{dx} = y^2$, $y(0) = \frac{1}{2}$ for $x > -1$.

This is a separable D.E.:

$\dfrac{dy}{y^2} = \dfrac{dx}{x+1}$

$\Rightarrow \int \dfrac{dy}{y^2} = \int \dfrac{dx}{x+1}$

$\Rightarrow -\dfrac{1}{y} = \ln|x+1| + C$

$\Rightarrow y = \dfrac{-1}{\ln|x+1|+C}$

or $y = \dfrac{-1}{\ln(x+1)+C}$ since $x > -1$

Since $y(0) = \frac{1}{2}$, then $\frac{1}{2} = \dfrac{-1}{\ln(0+1)+C} = -\dfrac{1}{C} \Rightarrow C = -2$

Thus $y = \dfrac{-1}{\ln(x+1)-2}$ or $\boxed{y = \dfrac{1}{2-\ln(x+1)}}$

continued ...

6. [6] A cup of coffee is poured from a pot whose contents are 95°C into a non-insulated cup in a room at 20°C. Let $T(t)$ be the temperature of the coffee after t minutes. Assuming that the coffee cools according to Newton's Law, then

$$\frac{dT}{dt} = k(T-20).$$

(a) Solve this differential equation subject to the initial condition $T(0) = 95$.

This is a separable D.E.!

$\frac{dT}{T-20} = k\,dt$ if $T \neq 20$

$\Rightarrow \int \frac{dT}{T-20} = \int k\,dt$

$\Rightarrow \ln|T-20| = kt+C$

$\Rightarrow \ln(T-20) = kt+C$, since we can assume that $T-20>0$

$\Rightarrow T-20 = e^{C} \cdot e^{kt}$

$\Rightarrow T = e^{C}e^{kt}+20$

Since $T(0)=95$, then $95 = e^{C}e^{k(0)}+20 \Rightarrow e^{C}=75$

Thus $T = 75e^{kt}+20$

(b) After one minute, the coffee has cooled to 90°C. Use this information and your solution to (a) to solve for the constant k.

$T(t) = 75e^{kt}+20$

$T(1) = 90 = 75e^{k}+20$

$\Rightarrow 70 = 75e^{k}$

$\Rightarrow \frac{70}{75} = e^{k}$

so $\boxed{k = \ln\left(\frac{70}{75}\right)}$

continued . . .

7. [6] Populations of aphids (A) and ladybugs (L) are modeled by the predator-prey equations

$$\frac{dA}{dt} = 2A - 0.01AL$$
$$\frac{dL}{dt} = -0.5L + 0.0001AL$$

(a) Find the equilibrium solutions and explain their significance.

Set $\frac{dA}{dt} = 0$ & $\frac{dL}{dt} = 0$.

$2A - 0.01AL = 0$
$\Rightarrow A(2 - 0.01L) = 0$
$\Rightarrow A = 0$ or
$L = 200$

$-0.5L + 0.0001AL = 0$
$\Rightarrow L(-0.5 + 0.0001A) = 0$
$\Rightarrow L = 0$ or $A = 5000$

The equilibria solutions are $A = L = 0$ or $A = 5000, L = 200$. At these values, the populations will remain unchanged over time.

(b) When there are 1000 aphids and 300 ladybugs, is the aphid population increasing or decreasing? Justify your answer.

$$\frac{dA}{dt} = 2A - 0.01AL$$
$$= 2(1000) - 0.01(1000)(300)$$
$$= 2000 - 3000$$
$$= -1000$$
$$< 0$$

So, the population will be decreasing since $\frac{dA}{dt}$ under these conditions.

continued ...

Name Solutions

Student Number ____________________

Your TA's Name: ____________________

Arts & Science 1D6

DAY CLASS
DECEMBER EXAM
DURATION OF EXAM: 2 Hours
MCMASTER UNIVERSITY

DR. MATT VALERIOTE

17 December, 2011

THIS EXAMINATION PAPER INCLUDES 12 PAGES AND 12 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF YOUR INVIGILATOR.

Attempt all questions.
The total number of available points is 50.
Marks are indicated next to each question.
Use of a Casio fx991 calculator only is allowed.
Write your answers in the space provided.
You must show your work to get full credit.
Use the last page for rough work.
Good Luck.

Score

Question	1–4	5	6	7	8
Points	8	6	6	4	3
Score					
Question	9	10	11	12	Total
Points	6	3	9	5	50
Score					

Continued on Page 2 ...

Multiple Choice & True/False Questions.

Indicate your answers to questions 1 and 2 by circling only ONE of the letters. You do not need to provide justifications for your answers to these two questions. Each of these questions is worth 2 marks.

1. [2] If $g(x) = (f(x))^3 + f(x^3)$, $f(1) = 2$, and $f'(1) = -1$, then $g'(1)$ is equal to:

(A) 0 (B) -4 (C) 4 (D) -12

(E) 12 (F) -24 (G) 24 [✓ (H) -15]

$g'(x) = [(f(x))^3]' + (f(x^3))'$

$= 3(f(x))^2 \cdot f'(x) + f'(x^3)(3x^2)$ [Chain Rule]

So $g'(1) = 3(f(1))^2 \cdot f'(1) + f'(1^3)(3(1)^2)$

$= 3(2)^2 \cdot (-1) + (-1)(3)$

$= -12 - 3 = \boxed{-15}$

2. [2] Find the constant(s) c that make(s) the following function continuous everywhere:

$$f(x) = \begin{cases} c^2 - x^2 & \text{if } x < 2 \\ 2(c - x) & \text{if } x \geq 2 \end{cases}.$$

(A) $-4, -2$ [✓ (B) 0, 2] (C) 2 (D) 4

(E) $-2, 4$ (F) -2 (G) 0 (H) Does not exist

$c^2 - x^2$ & $2(c-x)$ are continuous functions for all constants c, so f will be continuous precisely when $c^2 - (2)^2 = 2(c - (2))$

or when: $c^2 - 4 = 2c - 4$

$\Rightarrow c^2 = 2c$

$\Rightarrow c^2 - 2c = 0$ or $c(c-2) = 0$

$\Rightarrow c = 0$ or $c = 2$

Continued on Page 3...

Indicate your answers to questions 3 and 4 by circling only ONE of TRUE or FALSE. To receive credit for your solutions, you must justify your answers. Each of these questions is worth 2 marks.

3. [2] If c is a critical number of the function f, then $f'(c) = 0$.

TRUE [✓ FALSE]

f could have a critical number at c, if $f'(c)$ fails to exist. For example $c=0$ is a critical number of the function $f(x)=|x|$, but $f'(0) \neq 0$, since $f'(0)$ does not exist.

4. [2] If $g(x)$ is an even function that is continuous at all values, then $\int_{-1}^{1} xg(x)\,dx = 0$.

[✓ TRUE] FALSE

Since $g(x)$ is an even function, then $f(x)=xg(x)$ is an odd function [Proof: $f(-x)=(-x)g(-x)$ $=-xg(x)=-f(x)$].

In general, for an odd function f, $\int_{-a}^{a} f(x)\,dx=0$, since the area of the region bounded by $f(x)$ above the x-axis, for $x \geq 0$ is offset by the area of the region bounded by $f(x)$ below the x-axis, for $x \leq 0$.

Continued on Page 4 . . .

Questions 5–12: you must show work to receive full credit.

5. Let $f(x) = \ln\left(\frac{x}{x-3}\right)$.

(a) [3] What is the domain of $f(x)$?

$F(x)$ is defined when ① $\frac{x}{x-3}$ is defined & when ② $\frac{x}{x-3} > 0$.

So, $f(x)$ is defined when $x \neq 3$ & [when $(x>0$ & $x-3>0)$ or when $(x<0$ & $x-3<0)$]

So, the domain of f is $\{x \mid x>3 \text{ or } x<0\}$
$= (-\infty, 0) \cup (3, \infty)$

(b) [3] Find $f^{-1}(x)$, the inverse of the function $f(x)$. What is the domain of $f^{-1}(x)$ (and hence the range of $f(x)$)? [You do not need to show that $f(x)$ is one-to-one.]

① Set $y = f(x)$ & solve for x in terms of y:

$$y = \ln\left(\frac{x}{x-3}\right)$$
$$e^y = \frac{x}{x-3}$$
$$(x-3)e^y = x$$
$$x(e^y - 1) = 3e^y$$
$$x = \frac{3e^y}{e^y - 1}$$

② Interchange x & y:

$$y = \frac{3e^x}{e^x - 1}$$

③ $$f^{-1}(x) = \frac{3e^x}{e^x - 1}$$

The domain of $f^{-1}(x)$ consists of all x for which $e^x - 1 \neq 0$ or $e^x \neq 1$. Since $e^x = 1$, if and only if $x = 0$, then the domain of f^{-1} is $\{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$.

Continued on Page 5 . . .

6. Compute the following limits:

(a) [2] $\lim_{x\to\infty}\frac{1+x\sqrt{x}}{\sqrt{x}+\frac{1}{\sqrt{x}}}$. ← Indeterminate of form $\frac{\infty}{\infty}$.

Divide numerator & denominator by $\sqrt{x}$, the highest power of x in the denominator:

$$\lim_{x\to\infty}\frac{1+x\sqrt{x}}{\sqrt{x}+\frac{1}{\sqrt{x}}} = \lim_{x\to\infty}\frac{\frac{1}{\sqrt{x}}(1+x\sqrt{x})}{\frac{1}{\sqrt{x}}(\sqrt{x}+\frac{1}{\sqrt{x}})} = \lim_{x\to\infty}\frac{(\frac{1}{\sqrt{x}}+x)}{1+\frac{1}{x}} = \infty.$$

Note: L'Hospital's Rule could also be used to solve this limit.

(b) [2] $\lim_{h\to0}\frac{e^{5+2h}-e^5}{h}$. ← indeterminate of form $\frac{0}{0}$

$$\overset{LH}{=} \lim_{h\to0}\frac{(2)e^{5+2h}-0}{1} = \lim_{h\to0} 2e^{5+2h} = 2\cdot e^5$$

Note: If $f(x)=e^{5+2x}$, then $f'(0)=\lim_{h\to0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to0}\frac{e^{5+2h}-e^5}{h}$.

Since $f'(x)=2\cdot e^{5+2x}$, then $f'(0)=2\cdot e^{5+2(0)}=2e^5$

So, $\lim_{h\to0}\frac{e^{5+2h}e^5}{h}=2e^5$.

(c) [2] $\lim_{x\to0^+} x^{(1/x)}$. ← this limit has form 0^∞, which is determinate.

As $x\to0^+$, $x^{(\frac{1}{x})}$ approaches 0 & so

$\lim_{x\to0^+} x^{(1/x)}=0$.

⇒ Alternatively: Since $\lim_{x\to0^+}(\frac{1}{x})\ln(x) = -\infty$, then

$$\lim_{x\to0^+} x^{(1/x)} = \lim_{x\to0^+} e^{[\frac{1}{x}\ln(x)]} = e^{[\lim_{x\to0^+}(\frac{1}{x})\ln x]} = e^{-\infty} = 0.$$

Continued on Page 6 . . .

7. It is estimated that following the major earthquake that struck off of the coast of Japan earlier this year, 2,000 grams of the radioactive substance Cesium-137 were released into the atmosphere from the crippled Fukushima Daiichi Nuclear Power Plant. The half-life of Cesium-137 is 30 years, and so 30 years from now, one-half of the released material will remain.

(a) [2] Find a formula for $m(t)$, the mass of the remaining Cesium-137, after t years.

$m(t) = Ae^{kt}$ for some constants A & k.
$m(0) = Ae^{k(0)} = A$. We are given that $m(0) = 2000$ grams
So $A = 2000$.
Also, $m(30) = \frac{1}{2} m(0) = \frac{1}{2}(2000) = 1000$
so $1000 = 2000e^{k(30)}$
$\Rightarrow e^{30k} = \frac{1}{2} \Rightarrow 30k = \ln(\frac{1}{2}) = -\ln(2)$

$\Rightarrow k = -\frac{\ln(2)}{30}$.
So, $m(t) = 2000e^{-\frac{\ln(2)}{30}t}$ grams
$\boxed{= 2000 \cdot 2^{-\frac{t}{30}}}$ grams

(b) [2] How long will it take for the mass of the remaining Cesium-137 to be reduced to 100 grams?

Solve for t in $m(t) = 100$: $100 = 2000e^{\left(-\frac{\ln(2)}{30}t\right)} \Rightarrow e^{-\frac{\ln(2)}{30}t} = \frac{1}{20}$ or $e^{\frac{\ln(2)}{30}t} = 20$
$\Rightarrow \frac{\ln(2)}{30} t = \ln(20)$ or ~~$\frac{600}{\ln(2)}$ years~~
$t = \frac{30 \cdot \ln(20)}{\ln(2)} = 129.66$ years

8. [3] Let $g(x)$ be a function that is continuous on the interval $[2, 4]$. If $g(2) > 2$ and $g(4) < 4$ show that there is a solution to the equation $g(x) = x$ in the interval $(2, 4)$, i.e., there is some number c with $2 < c < 4$ and with $g(c) = c$.

Use the Intermediate Value Theorem!
Let $f(x) = g(x) - x$ & show there is some number c with $2 < c < 4$ with $f(c) = 0$.
Since $g(2) > 2$, then $g(2) - 2 > 0$, so $f(2) > 0$.
Since $g(4) < 4$, then $g(4) - 4 < 0$, so $f(4) < 0$.
Since g is continuous on $[2,4]$, then so is f &
thus by the IVT, there is some c in $(2,4)$
with $f(c) = 0$, or with $g(c) = c$

Continued on Page 7 . . .

9. Find the derivatives of the following functions. You do not need to simplify your answers.

(a) [3] $h(t) = \arcsin(t^2) - \dfrac{e^t}{1+e^{2t}}$.

$$h'(t) = \frac{1}{\sqrt{1-(t^2)^2}} \cdot (2t) - \frac{e^t(1+e^{2t}) - e^t(2e^{2t})}{(1+e^{2t})^2}$$

[Chain Rule & Quotient Rule]

$$= \frac{2t}{\sqrt{1-t^4}} - \frac{e^t - e^{3t}}{(1+e^{2t})^2}$$

(b) [3] $f(x) = \ln(x)\cos(\tan(x))$.

$$f'(x) = \frac{1}{x} \cdot \cos(\tan(x)) + \ln(x) \cdot \left(-\sin(\tan(x)) \cdot \sec^2(x)\right)$$

$$= \frac{\cos(\tan(x))}{x} - \ln(x)\sin(\tan(x))\sec^2(x).$$

10. [3] Find the function $f(x)$ that satisfies the given conditions:

$$f'(x) = (x-1)^3 + 2 + \frac{1}{1+x^2} \text{ and } f(1) = 2.$$

The most general anti-derivative of the given function is

$$f(x) = \frac{1}{4}(x-1)^4 + 2x + \arctan(x) + C$$

Using $f(1) = 2$, we can solve for C:

$$f(1) = \frac{1}{4}(1-1)^4 + 2(1) + \arctan(1) + C = 2$$

$$\Rightarrow 2 + \arctan(1) + C = 2$$

$$\Rightarrow C = -\arctan(1) = -\frac{\pi}{4}$$

So, $f(x) = \frac{1}{4}(x-1)^4 + 2x + \arctan(x) - \frac{\pi}{4}$

Continued on Page 8 . . .

11. Let $f(x) = 4x^2 - \frac{1}{x}$; then $f'(x) = 8x + \frac{1}{x^2}$ and $f''(x) = 8 - \frac{2}{x^3}$.

(a) [6] For the function f, find the domain, $x-$ and $y-$ intercepts, any symmetries, all asymptotes, all local extreme values and intervals of increase and decrease, all intervals where f is concave up and concave down, and inflection points. Place your answers in the following table. Use the next page for rough work.

ANSWERS:

domain of f: $x \neq 0$

$x-$intercept(s): $x = \frac{1}{\sqrt[3]{4}}$

$y-$intercept(s): ——

symmetries: None

horizontal asymptote(s): None

vertical asymptote(s): $x=0$. $\lim_{x\to0^+} f(x) = -\infty$, $\lim_{x\to0^-} f(x) = +\infty$

local extreme values (if any): local min at $x=-\frac{1}{2}$, $f(-\frac{1}{2})= 3$

f is increasing on: $x>0$ ~~f~~ or $-\frac{1}{2}<x<0$

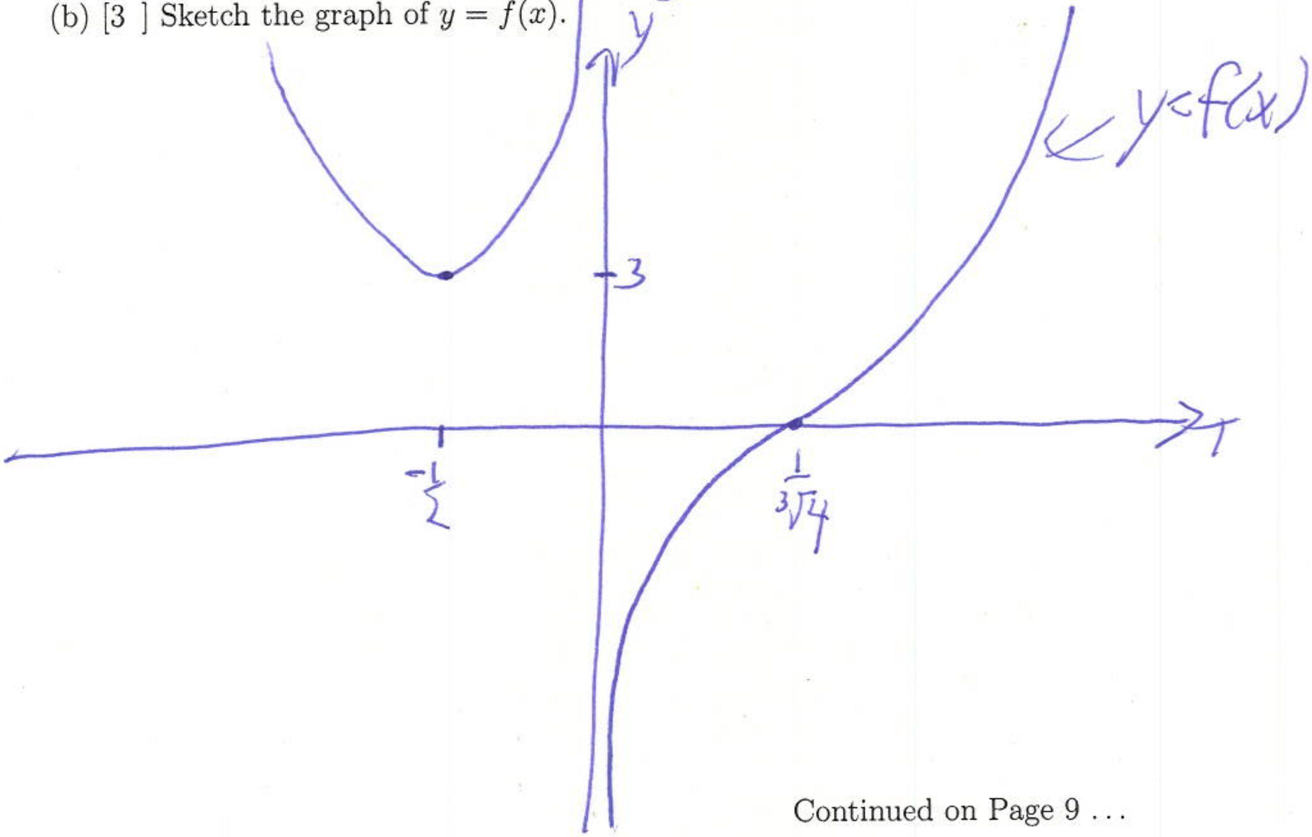
f is decreasing on: $x<-\frac{1}{2}$.

inflection points (if any): $x = \frac{1}{\sqrt[3]{4}}$

f is concave up on: $x > \frac{1}{\sqrt[3]{4}}$ or $x<0$

f is concave down on: $0<x<\frac{1}{\sqrt[3]{4}}$, $x<0$

(b) [3] Sketch the graph of $y = f(x)$.



Continued on Page 9 . . .

Space for rough work for question #11.

x-int: Solve $4x^2-\frac{1}{x}=0 \Rightarrow 4x^2=\frac{1}{x} \Rightarrow 4x^3=1 \Rightarrow x=\frac{1}{\sqrt[3]{4}}$

$\lim_{x\to 0^+} f(x)=-\infty$, $\lim_{x\to 0^-} f(x)=\infty$

$f'(x)= 8x+\frac{1}{x^2}$

$f'(x)=0$ when $8x+\frac{1}{x^2}=0$ or $8x=\frac{-1}{x^2}$ or $8x^3=-1$
$\Rightarrow x=-\frac{1}{2}$

$f'(x)>0$ when $x>0$ or when $-\frac{1}{2}<x<0$

$f'(x)<0$ when $x<-\frac{1}{2}$

So f has a local min at $x=-\frac{1}{2}$

$f''(x)=0$ when $8-\frac{2}{x^3}=0$ or $8x^3=2$ or $x^3=\frac{2}{8}$ or $x=\frac{1}{\sqrt[3]{4}}$

$f''(x)>0$ when $8-\frac{2}{x^3}>0$ or $8x^3>2$ or $x^3>\frac{1}{4}$ or $x>\frac{1}{\sqrt[3]{4}}$
$\underline{x>0}$
or $x<0$

$f''(x)<0$ when $0<x<\frac{1}{\sqrt[3]{4}}$

Continued on Page 10 . . .

12. (a) [3] Estimate $\int_1^3 (x - x^2)\,dx$ by evaluating the Riemann sum for $f(x) = x - x^2$, with $n = 4$, and by taking the sample points to be the midpoints of each subinterval.

$a=1, b=3, n=4, \Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$

$x_0 = a = 1, x_1 = a+\Delta x = \frac{3}{2}, x_2 = a+2\Delta x = 1+1=2, x_3 = a+3\Delta x = 1+\frac{3}{2} = \frac{5}{2}, x_4 = b = 3$

$M_4 = \sum_{i=1}^{4} f(\bar{x}_i)\Delta x$, where $\bar{x}_i = \frac{x_{i-1}+x_i}{2}$

$\bar{x}_1 = \frac{1+\frac{3}{2}}{2} = \frac{5}{4}, \bar{x}_2 = \frac{\frac{3}{2}+2}{2} = \frac{7}{4}, \bar{x}_3 = \frac{2+\frac{5}{2}}{2} = \frac{9}{4}, \bar{x}_4 = \frac{\frac{5}{2}+3}{2} = \frac{11}{4}$

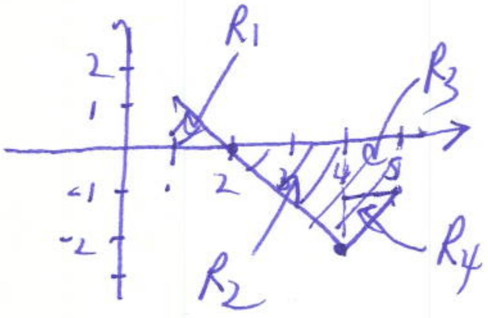
$M_4 = f(\frac{5}{4})(\frac{1}{2}) + f(\frac{7}{4})(\frac{1}{2}) + f(\frac{9}{4})(\frac{1}{2}) + f(\frac{11}{4})(\frac{1}{2})$

$= \frac{1}{2}[f(\frac{5}{4}) + f(\frac{7}{4}) + f(\frac{9}{4}) + f(\frac{11}{4})]$

$= \frac{1}{2}[(\frac{5}{4}+\frac{7}{4}+\frac{9}{4}+\frac{11}{4}) - ((\frac{5}{4})^2+(\frac{7}{4})^2+(\frac{9}{4})^2+(\frac{11}{4})^2)] = \frac{1}{2}[8 - \frac{276}{16}] = \frac{-37}{8} = -4.625.$

(b) [2] Find the exact value of the definite integral $\int_1^5 (|x-4|-2)\,dx$. (Hint: Interpret the definite integral in terms of net area.)

The graph of $y = |x-x|-2$ is :



Area of region above the x-axis is $(1)(1)\frac{1}{2} = \frac{1}{2}$

Area of region below the x-axis is: $(2)(2)(\frac{1}{2}) + (1)(1) + 1\cdot 1\cdot(\frac{1}{2})$

$= 2+1+\frac{1}{2} = 3\frac{1}{2}$

Net area $= (\frac{1}{2}) - (3\frac{1}{2})$

$\boxed{= -3}$

Continued on Page 11 ...

Arts & Science 1D06 Test #2

Day Class
Test #2
Duration of test: 60 minutes
McMaster University
28 February, 2012

Dr. Matt Valeriote

Last Name: Solutions

Initials:_________

Student No.:__________________________

Your TA's Name:____________

This test has 8 pages and 7 questions and is printed on BOTH sides of the paper. Pages 7 and 8 contain no questions and can be used for rough work.

You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancies to the attention of the invigilator.

Attempt all questions and write your answers in the space provided.

Marks are indicated next to each question; the total number of marks is 40.

Any Casio fx991 calculator is allowed. Other aids are not permitted.

Use pen to write your test. If you use a pencil, your test will not be accepted for regrading (if needed).

Good Luck!

Score

Question	1	2	3	4	5	6	7	Total
Points	7	6	5	4	6	6	6	40
Score								

continued . . .

ALL QUESTIONS: you must show your work to receive full credit.

1(a) [4] State both parts of the Fundamental Theorem of Calculus

Suppose that f is continuous on $[a,b]$.

① If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$

② $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, i.e. $F' = f$.

(b) [3] Use the Fundamental Theorem of Calculus to find the derivative of the function

$$y = \int_{\sin x}^{\cos x} (1 + v^2)^{10}\, dv.$$

⇒ We also need to use the Chain Rule

$$y = \int_{\sin x}^{\cos x} (1+v^2)^{10}\,dv = \int_{\sin x}^{0} (1+v^2)^{10}\,dv + \int_{0}^{\cos x} (1+v^2)^{10}\,dv$$

$$= -\int_{0}^{\sin x} (1+v^2)^{10}\,dv + \int_{0}^{\cos x} (1+v^2)^{10}\,dv,$$

$$\text{so } y' = \left[-\int_0^{\sin x} (1+v^2)^{10}\,dv\right]' + \left[\int_0^{\cos x} (1+v^2)^{10}\,dv\right]'$$

$$= -(1+\sin^3 x)^{10}(\sin x)' + (1+\cos^3 x)^{10}(\cos x)'$$

$$= -(1+\sin^3 x)^{10}(\cos x) - (1+\cos^3 x)^{10}(\sin x).$$

continued . . .

2. [6] Find $\int \sin\sqrt{x}\,dx$. [Hint: After making a suitable substitution, use integration by parts.]

Let $s=\sqrt{x}$. Then $ds=\frac{1}{2\sqrt{x}}dx=\frac{1}{2s}dx$

so $\int \sin\sqrt{x}dx=\int \sin(s)\cdot 2s\,ds \;=\int 2s\cdot\sin(s)ds.$

Use Int. by Parts with $u=2s$, $dv=\sin(s)ds$

$du=2ds \quad v=-\cos(s)$

$$\text{so} \int 2s\cdot\sin(s)ds = uv-\int vdu = (2s)(-\cos(s))-\int(-\cos(s))\cdot 2ds$$

$$=-2s\cos(s)+2\int\cos(s)ds$$

$$=-2s\cos(s)+2\sin(s)+C$$

$$=-2\sqrt{x}\cdot\cos(\sqrt{x})+2\sin(\sqrt{x})+C$$

3. [5] Let $p>1$. Show that the following improper integral is convergent and evaluate it.

$$\int_1^\infty \frac{1}{x^p}\,dx.$$

$$\int_1^\infty \frac{1}{x^p}dx=\lim_{t\to\infty}\int_1^t\frac{1}{x^p}dx$$

$$=\lim_{t\to\infty}\left[\frac{1}{-p+1}\cdot\frac{1}{x^{p-1}}\Big|_1^t\right]$$

$$=\lim_{t\to\infty}\left[\frac{1}{1-p}\cdot\frac{1}{t^{p-1}}-\frac{1}{1-p}\cdot\frac{1}{1^{p-1}}\right]$$

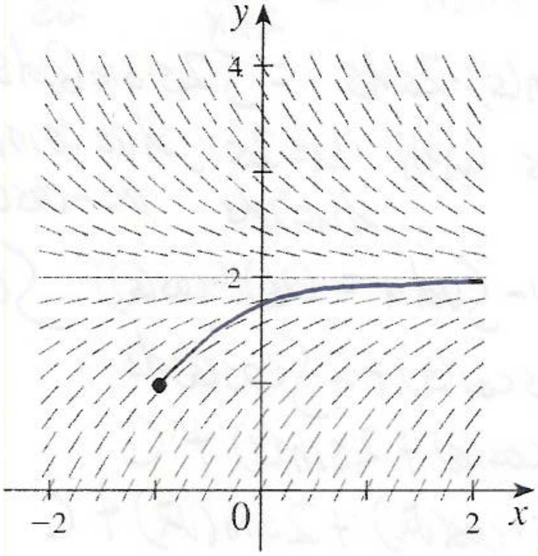
$$=\left(\frac{1}{1-p}\cdot\lim_{t\to\infty}\frac{1}{t^{p-1}}\right)-\frac{1}{1-p}$$

$=0+\frac{1}{p-1}$, since $\lim_{t\to\infty}\frac{1}{t^{p-1}}=0$ when $p>1$

$$\boxed{=\frac{1}{p-1}}$$

continued . . .

4. [4] A direction field for a particular differential equation is given below.



(a) On the direction field, sketch the graph of the solution to the differential equation that passes through the point $(-1, 1)$.

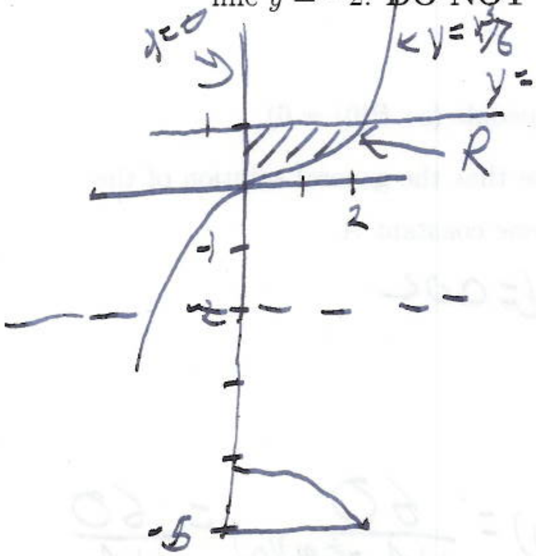
(b) Which of the following differential equations matches the given direction field? Circle your answer.

[✓ (A) $\frac{dy}{dx} = 2 - y$] (B) $\frac{dy}{dx} = x^2 - y^2$ (C) $\frac{dy}{dx} = y - 1$ (D) $\frac{dy}{dx} = x^2 + 2$

④ Observe: that the slopes of the tangent lines in the Direction Field do not depend on the value of x. This rules out (B) & (D).
- Also when y = 2, the slope of the tangent lines are 0. This rules out (C) & so the answer is (A).

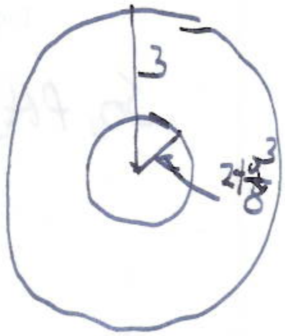
continued . . .

5. [6] Let R be the region bounded by $x = 0$, $y = 1$, and the curve $y = \frac{x^3}{8}$. Sketch the region R and set up an integral for the volume of the solid obtained by rotating R about the horizontal line $y = -2$. **DO NOT EVALUATE THE INTEGRAL!**



$$V = \int_0^2 \left[\pi(3)^2 - \pi\left(2+\frac{x^3}{8}\right)^2\right] dx$$

$$= \pi \int_0^2 \left[9 - \left(2+\frac{x^3}{8}\right)^2\right] dx$$



6. [6] Solve the differential equation: $\frac{dy}{dx} = \frac{xy + 3x}{x^2 + 1}$.

This is a seperable first order d.e.!

$\frac{dy}{dx} = \frac{x(y+3)}{x^2+1}$ ⊛

$\Rightarrow \frac{dy}{y+3} = \frac{x}{x^2+1}dx$, if $y+3 \neq 0$, or $y \neq -3$

$\left[\text{let } u = x^2+1,\ du = 2x\,dx \quad \int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C\right]$

$\Rightarrow \int \frac{dy}{y+3} = \int \frac{x}{x^2+1}dx$

$\Rightarrow \ln|y+3| = \frac{1}{2}\ln(x^2+1) + C$

or $\ln|y+3| = \ln\left(\sqrt{x^2+1}\right) + C$

$\Rightarrow |y+3| = e^{\ln(\sqrt{x^2+1})} \cdot e^C$

$\Rightarrow y+3 = \pm e^C \sqrt{x^2+1}$

$y = -3 \pm e^C \sqrt{x^2+1}$

$\Rightarrow y = -3 + A\sqrt{x^2+1}$, A any nonzero constant.

⊛ But: $y=-3$ is also a solution, since $\frac{dy}{dx} = 0 = \frac{x(-3)+3x}{x^2+1}$.

So, the general solution is $y = -3 + A\sqrt{x^2+1}$ for any constant A.

continued ...

7. [6] Let $P(t)$ be the population of the Earth t years after the year 2000 and assume that $P(t)$ grows according to the logistic equation

$$\frac{dP}{dt} = (0.02)P\left(1 - \frac{P}{60}\right).$$

In the year 2000 the population of the Earth was 6 billion people (so $P(0) = 6$).

(a) Write a formula for the population $P(t)$. You may use that the general solution of the logistic equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ is $\frac{M}{1 + Ae^{-kt}}$ for some constant A.

So, $P(t) = \frac{M}{1+Ae^{-kt}}$, where $M = 60$ & $k = 0.02$

$= \frac{60}{1+Ae^{-(0.02)t}}$

To solve for A: use $P(0) = 6$ & $P(0) = \frac{60}{1+Ae^{-(0.02)(0)}} = \frac{60}{1+A}$

$\Rightarrow 6 = \frac{60}{1+A} \Rightarrow 1+A = \frac{60}{6} = 10 \Rightarrow \underline{A = 9}$ [or use: $A = \frac{M-P_0}{P_0}$]

Thus $P(t) = \frac{60}{1+9e^{-0.02t}}$

(b) What is the projected population of the Earth in the year 2100?

The population in the year 2100 is equal to

$P(100) = \frac{60}{1+9e^{-0.02(100)}} = \frac{60}{1+9(0.13533..)} = 27.05$

So, according to this model, the population of the Earth in 2100 will be 27 billion.

continued . . .

Name ____________________

Student Number ____________________

Your TA's Name: ____________________

Arts & Science 1D06

DAY CLASS
APRIL EXAM
DURATION OF EXAM: 3 Hours
MCMASTER UNIVERSITY

DR. MATT VALERIOTE

10 April, 2012

THIS EXAMINATION PAPER INCLUDES 14 PAGES AND 17 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF YOUR INVIGILATOR.

Attempt all questions.
The total number of available points is 100.
Marks are indicated next to each question.
Use of a Casio fx991 calculator only is allowed.
Write your answers in the space provided.
You must show your work to get full credit.
Use the last two pages for rough work.
Good Luck.

Score

Question	1–3	4–6	7	8	9	10	11
Points	9	9	4	10	8	6	7
Score							

Question	12	13	14	15	16	17	Total
Points	6	10	8	8	5	10	100
Score							

Continued on Page 2 . . .

Multiple Choice Questions

Indicate your answers to questions 1–3 by circling only ONE of the letters. Each of these questions is worth 3 marks.

1. [3] Let

$$f(x) = \frac{1}{e^x + 1}.$$

Which of the following statements are **true**?

(I) The domain of $f(x)$ is $(-\infty, \infty)$. TRUE

(II) $f(x)$ is an odd function. FALSE

(III) $f(x)$ has an inverse. TRUE

(A) none
(B) I only
(C) II only
(D) III only
(E) I and II
[✓ (F) I and III]
(G) II and III
(H) all three

$y = \frac{1}{e^x+1} \Rightarrow \frac{1}{y} = e^x + 1 \Rightarrow e^x = \frac{1}{y} - 1 \Rightarrow x = \ln(\frac{1}{y} - 1)$

So, $f^{-1}(x) = \ln(\frac{1}{x} - 1)$

2. [3] Which of the following series are convergent?

$$\text{(I)} \sum_{n=1}^{\infty} (-1)^n \qquad \text{(II)} \sum_{n=1}^{\infty} 2^n \qquad \text{(III)} \sum_{n=1}^{\infty} \frac{1}{2+n^3}$$

(A) none
(B) I only
(C) II only
[✓ (D) III only]
(E) I and II
(F) I and III
(G) II and III
(H) all three

$\sum_{n=1}^{\infty} (-1)^n$ is divergent, since $\lim_{n\to\infty} (-1)^n \neq 0$ (Test for Divergence)

$\sum_{n=1}^{\infty} 2^n$ is divergent, since $\lim_{n\to\infty} 2^n \neq 0$ (Test for Divergence)

$\sum_{n=1}^{\infty} \frac{1}{2+n^3}$ is convergent, by comparison with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Continued on Page 3...

3. [3] Let $g(x) = \int_{x^2}^{1} \sin(\sqrt{t})\,dt$. Then $g'(\pi/2)$ is:

(A) 0 [✓ (B) $-\pi$]

(C) $\sin(1) - 1$ (D) -1

$g(x) = -\int_{1}^{x^2} \sin(\sqrt{t})dt$, so by th FTC & Chain Rule,

$g'(x) = -\sin(\sqrt{x^2})(x^2)'$

$= -\sin(x)(2x)$ (if $x>0$)

$= -2x\sin(x)$

so $g'(\frac{\pi}{2}) = -2(\frac{\pi}{2})\sin(\frac{\pi}{2}) = -\pi(1) = -\pi$

True/False Questions.

Decide whether the statements in questions 4–6 are true or false by circling your choice. YOU MUST JUSTIFY YOUR ANSWER TO RECEIVE FULL CREDIT. Each of these questions is worth 3 marks.

4. [3] There is some value of x in the interval $(2, 3)$ such that $x^3 - 5x - 7 = 0$.

[✓ TRUE] FALSE

Use the Intermediate Value Theorem:

Let $f(x) = x^3 - 5x - 7$.

Then f is continuous on $[2,3]$ since it is a polynomial,

$f(2) = 2^3 - 5(2) - 7 = 8 - 10 - 7 = -9 < 0$

& $f(3) = 3^3 - 5(3) - 7$

$= 27 - 15 - 7 = 5 > 0$

So, by the IVT there is some x with $2 < x < 3$

& $f(x) = 0$, i.e.,

$x^3 - 5x - 7 = 0$

Continued on Page 4…

5. [3] The improper integral $\int_0^\infty \cos(x)\,dx$ is convergent.

TRUE [✓ FALSE]

$\int_0^\infty \cos(x)dx = \lim_{t\to\infty}\int_0^t \cos(x)dx = \lim_{t\to\infty} \sin(x)\Big|_0^t = \lim_{t\to\infty}(\sin(t)-\sin(0))$

$= \lim_{t\to\infty}\sin(t)$ ← Does Not Exist.

So, the integral is divergent.

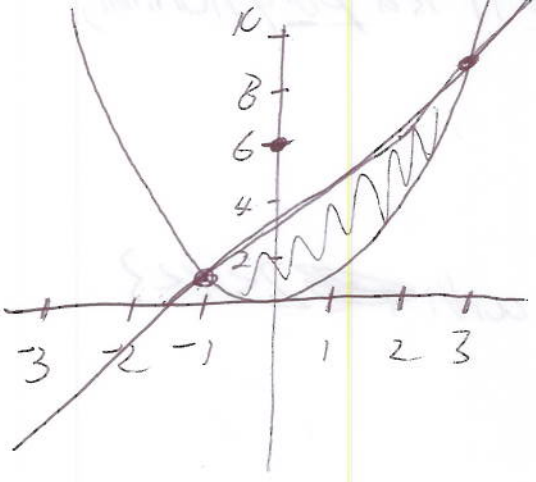
6. [3] If the power series $\sum_{n=1}^{\infty} c_n x^n$ converges when $x=-8$ then the series $\sum_{n=1}^{\infty} c_n 7^n$ converges.

[✓ TRUE] FALSE

Since the power series converges ~~in~~ when $x=-8$, then ~~the~~ R, the radius of convergence of the power series is ≥ 8. So, 7 is in the interval of convergence of the ~~po~~ power series so $\sum_{n=1}^{\infty} c_n 7^n$ is convergent.

Questions 7–17: you must show work to receive full credit

7. [4] Sketch the region bounded by the curves $y=x^2$ and $y=2x+3$ and set up an integral for its area. **Do not evaluate the integral!**



To find the points of intersection, solve

$x^2=2x+3 \Rightarrow x^2-2x-3=0$

$\Rightarrow (x-3)(x+1)=0 \Rightarrow x=3$ or $x=-1$

Area $= \int_{-1}^{3}(2x+3)-x^2\,dx$

$= \int_{-1}^{3}(2x+3-x^2)\,dx$

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8.(a) [5] Let $f(x) = \dfrac{x}{1+x^2} + 1$. Find the absolute maximum and absolute minimum values of $f(x)$ on the interval $[-3, 2]$.

Since f is continuous on $[-3,2]$, then by the Extreme Value Theorem, the absolute maximum & minimum will occur at the endpoints, -3 or 2, or at a critical point.

To find the critical points, solve $f'(x)=0$: $f'(x) = \dfrac{(1+x^2)-x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$.

So $f'(x)=0$ when $x^2=1$ or $x=\pm 1$.

Evaluate f at $-3,-1,1$ & 2: $f(-3) = \frac{-3}{10}+1 = \boxed{0.7}$, $f(-1) = \frac{-1}{1+1^2}+1 = \boxed{0.5}$

$f(1) = \frac{1}{1+1^2}+1 = \boxed{1.5}$, $f(2) = \frac{2}{1+2^2}+1 = 1.4$

The largest value is 1.5 & so the absolute max. value of f on $[-3,2]$ is 1.5

The smallest value is 0.5 & so the absolute min value of f on $[-3,2]$ is 0.5.

(b) [5] Find the intervals where $f(x) = \sin(2x) - 4\sin(x)$, $0 \le x \le \pi$, is concave up and concave down and identify all points of inflection.

Find where $f''(x)$ is <0, >0, or $=0$:

$f'(x) = 2\cos(2x) - 4\cos(x)$, $f''(x) = -4\sin(2x) + 4\sin(x)$

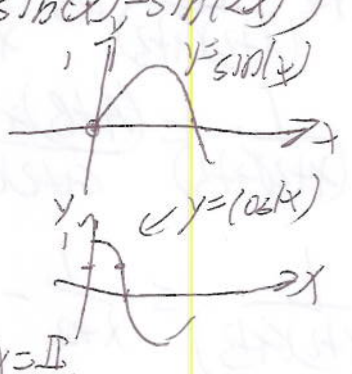
$= 4(\sin(x) - \sin(2x))$

Solve $f''(x)=0$: $4(\sin(x)-\sin(2x)) = 0$

$\Rightarrow \sin(x) - 2\sin(x)\cos(x) = 0$

$\Rightarrow \sin(x)[1-2\cos(x)] = 0$

$\Rightarrow \sin(x) = 0$, or $\cos(x) = \frac{1}{2}$

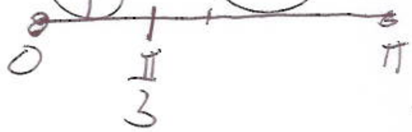


on $[0,\pi]$, this implies $x=0$ or $x=\pi$, or $x=\frac{\pi}{3}$.

So, on $[0,\pi]$, $f''(x)=0$ when $x=0, \frac{\pi}{3}$, or π.

In the interval $(0,\frac{\pi}{3})$, $f''(x) < 0$ (since $f''(\frac{\pi}{4}) < 0$)

& in the interval $(\frac{\pi}{3},\pi)$, $f''(x) > 0$ (since $f''(\frac{\pi}{2}) = 1$)



So f is Concave Down on $(0,\frac{\pi}{3})$ &

Concave up on $(\frac{\pi}{3},\pi)$.

f has points of inflection at $x=0, \frac{\pi}{3}$ & π.

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9. Compute the following integrals.

(a) [4] $\int_0^1 (x+1)e^{-x}\,dx$. Use Integration by Parts;

$u=(x+1)$, $dv=e^{-x}dx$

$du=dx$ $v=-e^{-x}$

$$\int_0^1 (x+1)e^{-x}dx = (x+1)(-e^{-x})\Big|_0^1 - \int_0^1 (-e^{-x})dx$$

$$=\left[(1+1)(-e^{-1})-(0+1)(-e^{-0})\right]+(-e^{-x})\Big|_0^1$$

$$=\left(-\frac{2}{e}+1\right)+\left[-e^{-1}+e^{0}\right]$$

$$=-\frac{2}{e}+1-\frac{1}{e}+1$$

$$=2-\frac{3}{e}$$

(b) [4] $\int \frac{1}{(x+2)(x+3)}\,dx$. ← Use Partial Fractions Method.

Write $\frac{1}{(x+2)(x+3)}$ as $\frac{A}{x+2}+\frac{B}{x+3}=\frac{A(x+3)+B(x+2)}{(x+2)(x+3)}=\frac{(A+B)x+(3A+2B)}{(x+2)(x+3)}$

So $\frac{1}{(x+2)(x+3)}=\frac{(A+B)x+(3A+2B)}{(x+2)(x+3)} \Rightarrow \begin{matrix} A+B=0 \\ 3A+2B=1 \end{matrix} \Rightarrow B=-A \Rightarrow 3A-2A=1$

$\Rightarrow A=1$

$\Rightarrow \underline{B=-1}$

So $\frac{1}{(x+2)(x+3)}=\frac{1}{x+2}-\frac{1}{x+3}$

Thus $\int\frac{1}{(x+2)(x+3)}dx=\int\frac{1}{x+2}dx-\int\frac{1}{x+3}dx$

$$=\ln|x+2|-\ln|x+3|+C$$

$$=\ln\left|\frac{x+2}{x+3}\right|+C$$

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10. [6] Consider the predator-prey system $x' = 4x - xy$, $y' = -y + \frac{xy}{2}$.

(a) Which of the variables, x or y, represents the predator? Explain why.

y represents the predator, since the term $+\frac{xy}{2}$ indicates that the y population will increase when there are interactions between the two species. The term $-xy$ in the first equation indicates that the x population will decrease when there are interactions.

(b) For each of the species represented by x and y, explain what happens if the other is not present.

when $y=0$, $x'=4x$ & so x will grow exponentially, $x(t)=x_0 e^{4t}$.

when $x=0$, $y'=-y$ & so y will decrease exponentially & $y(t)=y_0 e^{-t}$.

(c) Find all equilibrium solutions of this system.

Solve $x'=0$ & $y'=0$: $4x-xy=0 \Rightarrow x(4-y)=0 \Rightarrow x=0$ or $y=4$

& $-y+\frac{xy}{2}=0 \Rightarrow y(-1+\frac{x}{2})=0 \Rightarrow y=0$ or $x=2$

So, the equilibrium solutions are $x=0$ & $y=0$, & $x=2$ & $y=4$.

11.[7] Find the solution of the differential equation $\dfrac{dy}{dx} = \dfrac{(x^2 - x)}{e^y}$, that satisfies the initial condition $y(0) = 1$.

This is a separable d.e.: $e^y dy = (x^2-x)dx$

$\Rightarrow \int e^y dy = \int (x^2-x)dx \Rightarrow e^y = \frac{x^3}{3} - \frac{x^2}{2} + c \Rightarrow \underline{y = \ln(\frac{x^3}{3} - \frac{x^2}{2} + c)}$

since $y(0)=1$, then

$1 = \ln(\frac{0^3}{3} - \frac{0^2}{2} + c) \Rightarrow 1 = \ln(c) \Rightarrow \underline{c=e}$

So the solution is $y = \ln(\frac{x^3}{3} - \frac{x^2}{2} + e)$.

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12. [6] Suppose that the bowl of candy in C-105 initially contains 100 pieces and let $y = y(t)$ stand for the number of pieces of candy in the bowl after t hours.

(a) Find an exact expression for $y(t)$ assuming that one-third of the pieces in the bowl are removed each hour, and so y satisfies the differential equation $\frac{dy}{dt} = -\frac{y}{3}$.

The function y will decrease according to the law of natural decay & so will have the form $y(t) = Ae^{-t/3}$,

Since $y(0)=100$ & $y(0)=Ae^0=A$, then $A=100$,

so $y(t) = 100e^{-t/3}$.

(b) Now assume that Shelley is also continuously re-supplying the bowl at a rate of $\frac{75}{y}$ pieces per hour. Write a new differential equation that y satisfies in this case.

Since y is increased by $\frac{75}{y}$ pieces per hour, ~~&~~ (continuously) then y satisfies the d.e. $\frac{dy}{dt} = -\frac{y}{3} + \frac{75}{y}$

(c) Find the equilibrium amount of candy in the bowl in this situation.

Solve $\frac{dy}{dt}=0$: $-\frac{y}{3}+\frac{75}{y}=0$

$\Rightarrow$ [illegible] $\Rightarrow \frac{75}{y} = \frac{y}{3}$

$\Rightarrow 3\cdot 75 = y^2$

$\Rightarrow$ [illegible] $\Rightarrow y = \pm 15$

$\Rightarrow$ [illegible] $\Rightarrow \boxed{y = 15}$ since in this situation $y=-15$ is not allowed.

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13. Compute the following limits, or show that they do not exist. Justify your answers.

(a) [3] $\lim_{n\to\infty} \frac{e^n}{n!}$

$$\frac{e^n}{n!} = \frac{e\cdot e\cdot e\cdot e \cdots \cdot e\cdot e}{1\cdot 2\cdot 3\cdot 4 \cdots (n-1)(n)}$$

$$< \left(\frac{e}{1}\right)\left(\frac{e}{2}\right)\left(\frac{e}{n}\right) \text{ since for } k\geq 3 \; \frac{e}{k}<1$$

$$= \left(\frac{e^3}{2}\right)\cdot\frac{1}{n}$$

So $0 \leq \frac{e^n}{n!} \leq \frac{e^3}{2}\left(\frac{1}{n}\right)$. Since $\lim_{n\to\infty}\frac{e^3}{2}\left(\frac{1}{n}\right)=0$, then by the Squeeze Theorem,

$\lim_{n\to\infty}\frac{e^n}{n!}=0.$

(b) [3] $\lim_{n\to\infty} \frac{\ln(3n)}{\ln(n)} = \lim_{x\to\infty}\frac{\ln(3x)}{\ln(x)} \overset{L'H.}{=} \lim_{x\to\infty}\frac{\frac{1}{3x}(3)}{\frac{1}{2x}(2)} = 1$

So $\lim_{n\to\infty}\frac{\ln(3n)}{\ln(n)} = 1$

indeterminate of type $\infty - \infty$

(c) [4] $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin(x)}\right)$.

$= \lim_{x\to0}\frac{\sin(x)-x}{x\sin(x)} \leftarrow$ indeterminate of type $\frac{0}{0}$

$\overset{L'H.}{=} \lim_{x\to0}\frac{\cos(x)-1}{\sin(x)+x\cos(x)}$

$\overset{L'H.}{=} \lim_{x\to0}\frac{-\sin(x)}{\cos(x)+\cos(x)-x\sin(x)}$

$= \frac{0}{1+1-0} = 0$

So $\lim_{x\to0}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right)=0$

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14.(a) [2] Define the term "absolute convergence" for a series $\sum_{n=1}^{\infty} a_n$.

The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

(b) [3] Give an example of a series that is convergent, but not absolutely convergent.

The series $\sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{1}{n}\right)$ is convergent. This can be shown by using the Alternating Series Test. The series $\sum_{n=1}^{\infty} \left|(-1)^{n-1}\left(\frac{1}{n}\right)\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is the Harmonic Series & so is divergent. Thus $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ is convergent, but not absolutely convergent.

(c) [3] Determine if the series $\sum_{n=1}^{\infty} \frac{(-7)^n}{n6^n}$ is absolutely convergent.

Use the Ratio Test:

$$a_n = \frac{(-7)^n}{n6^n} \qquad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-7)^{n+1}}{(n+1)6^{n+1}}}{\frac{(-7)^n}{n6^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-7)^{n+1}}{(-7)^n}\cdot\frac{6^n}{6^{n+1}}\cdot\frac{n}{n+1}\right| = \lim_{n\to\infty}\frac{7}{6}\left(\frac{n}{n+1}\right) = \frac{7}{6}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)$$

$$= \frac{7}{6}.$$

Since $\frac{7}{8} > 1$, then by the Ratio Test this series is divergent & so is <u>not</u> absolutely convergent.

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15. Consider the power series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2 2^n}$.

(a) [4] Determine the radius of convergence of the power series.

Compute $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-5)^{n+1}}{(n+1)^2 2^{n+1}}}{\frac{(x-5)^n}{n^2 2^n}}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(x-5)^n}\cdot\frac{2^n}{2^{n+1}}\cdot\frac{n^2}{(n+1)^2}\right|$

$= \lim_{n\to\infty}\frac{|x-5|}{2}\frac{n^2}{(n+1)^2} = \frac{|x-5|}{2}$. $\frac{|x-5|}{2}<1$ when $|x-5|<2$

So the Radius of convergence is $R=2$

(b) [4] Determine the interval of convergence of the power series.

Check for convergence at $x=5+2=7$ & $x=5-2=3$:

$x=7$: $\sum_{n=1}^{\infty}\frac{(7-5)^n}{n^2 2^n} = \sum_{n=1}^{\infty}\frac{2^n}{n^2 2^n} = \sum_{n=1}^{\infty}\frac{1}{n^2}$ $\leftarrow$ a convergent p-series.

$x=3$: $\sum_{n=1}^{\infty}\frac{(3-5)^n}{n^2 2^n} = \sum_{n=1}^{\infty}\frac{(-2)^n}{n^2 2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ $\leftarrow$ a convergent alternating series.

So, the interval of convergence is $[3,7]$.

16. [5] Find the first four terms of the Maclaurin series for $f(x) = \frac{1}{\sqrt{1+2x}}$. $= (1+2x)^{-1/2}$

$\Rightarrow$ The Maclaurin series is $f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{3!}x^3+\cdots$

$f(0)=\frac{1}{\sqrt{1+0}}=1$, $f'(x)=\frac{-1}{2}(1+2x)^{-3/2}(2)$, so $f'(0)=-1$

$= -(1+2x)^{-3/2}$

$f''(x)=(-\frac{3}{2})(-1)(1+2x)^{-5/2}(2) \Rightarrow f''(0)=3$

$= 3(1+2x)^{-5/2}$

$f'''(x)=3(-\frac{5}{2})(1+2x)^{-7/2}(2)$ so $f'''(0)=-15$

$= -15(1+2x)^{-7/2}$

Thus the Maclaurin series for $f(x)$ starts as:

$1-x+\frac{3}{2}x^2-\frac{15}{6}x^3+\cdots$

$=1-x+\frac{3}{2}x^2-\frac{5}{2}x^3+\cdots$

* The Binomial Theorem could also be used

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17. (a) [3] State the Maclaurin series of the function $f(x) = \cos x$. You do not need to derive the series.

The Maclaurin series for cos(x) is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots,$$

(with radius of convergence $R = \infty$)

(b) [3]Find the Maclaurin series for the function $g(x) = \dfrac{1-\cos(x)}{x^2}$. Hint: Use your answer from (a).

The Maclaurin Series for g(x) is $\dfrac{\left(1 - \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right)}{x^2}$

$$= \frac{1}{x^2}\left(1 - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots + \frac{(-1)^{n-1} x^{2n}}{(2n)!} + \cdots\right)$$

$$= \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \cdots + \frac{(-1)^{n-1} x^{2n-2}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+2)!}$$

(c) [2] Use your answer from (b) to express $\int_0^1 \frac{1-\cos(x)}{x^2}\,dx$ as the sum of a series.

$$\int_0^1 \frac{1-\cos(x)}{x^2}dx = \int_0^1 \left(\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \cdots\right)dx$$

$$= \left(\frac{x}{2!} - \frac{x^3}{3\cdot 4!} + \frac{x^5}{5\cdot 6!} - \frac{x^7}{7\cdot 8!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+2)!} + \cdots\right)\Bigg|_0^1$$

$$= \frac{1}{2!} - \frac{1}{3\cdot 4!} + \frac{1}{5\cdot 6!} - \frac{1}{7\cdot 8!} + \cdots + \frac{(-1)^n}{(2n+1)(2n+2)!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)!}$$

(d) [2] The sum of the first two terms of the series from (c) provides an approximation of the definite integral from (c). Give an estimate for the error of this approximation.

Since the series from (c) satisfies the conditions of the Alternating Series Test then the third term in the series, $\frac{1}{5\cdot 6!}$ provides an estimate for the error given by $S_2 \doteq \frac{1}{2!} - \frac{1}{3\cdot 4!}$, the sum of the first two terms.

$$\frac{1}{5\cdot 6!} = \frac{1}{5\cdot 720} = \frac{1}{3600} = 0.0002777\cdots$$

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