N.B. You should also do questions 7 and 8 from assignment #7.

Name: ____________________

Arts Sci 1D06 Winter 2012 Assignment 13

Material covered: Sections 7.1, 7.2, 7.3, 7.4.

1. Show that the function $y = \frac{1}{2}(e^x + e^{-x})$ satisfies the differential equation $y'' = \sqrt{1 + (y')^2}$. Does $y = \frac{1}{2}(e^x - e^{-x})$ satisfy the same equation?

Continued on next page

2. Describe the following events as initial value problems (i.e., in each case write down a differential equation and an initial condition). Do not solve the equations.
(a) Ice starts forming at time $t = 0$. Let $T(t)$ be the thickness of the ice at time t. The rate at which ice is formed is inversely proportional to the square of its thickness.

(b) At time $t = 0$ somebody starts spreading the rumour that McMaster campus has been attacked by the Borg. Assume that there are $10,000$ students on the campus, and denote by $S(t)$ the number of people who have heard the rumour at time t. The rate of increase in the number of people who have heard the rumour is proportional to the number of people who have heard it and to the number of people who haven't heard it yet.

(c) A pie, initially at the temperature of 20^oC, is put into an 300^oC oven. Let $T(t)$ be the temperature of the pie at time t. The temperature of the pie changes proportionally to the difference between the temperature of the oven and the temperature of the pie.

Continued on next page

3. Solve the initial value problem $y' = \dfrac{1}{x^2y - 2x^2 + y - 2}$, $\qquad y(0) = 1.$

Continued on next page

4. Consider the initial value problem $y' = (y^2 + 1)x, \quad y(0) = 1$. Answer questions (a) and (b) WITHOUT SOLVING THE EQUATION:
(a) Find the intervals where the solution y is increasing and decreasing.

(b) Find all relative extreme values of y.

(c) Solve the given equation algebraically.

Continued on next page

5. The equation $2xyy' = y^2 - x^2$ is not separable. Show that, by introducing the new function $v = y/x$, the above equation can be reduced to a separable equation. Solve that equation, thus solving the original equation.

6. Repeat the previous exercise for the equation $xy' \sin(y/x) = y \sin(y/x) - x$.

Continued on next page

7. A population is modeled by the differential equation

$$\frac{dP(t)}{dt} = 1.42P(t)\left(1 - \frac{P(t)}{5600}\right).$$

(a) For what values of $P(t)$ is the population increasing? Decreasing?

(b) Explain what is an equilibrium solution. What are the equilibrium solutions of the given equation?

(c) Sketch the solutions of the given equation with initial conditions $P(0) = 2000$ and $P(0) = 10000$.

THE END

Name: ____________________

Arts Sci 1D06	**Winter 2012**	**Assignment 14**

Material covered: Sections 7.4, 7.5, 7.6.

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1. Solve the initial value problem $y' - 2xy = x$, $y(0) = 2$.

2. Solve the initial value problem $y' \cos\theta - y \sin\theta = \cos\theta$, $-\pi/2 \le \theta \le \pi/2$, $y(0) = 1$.

Continued on next page

3. The change in populations of red-footed foxes and white-tailed brown rabbits can be described by the following set of equations: $\frac{dx}{dt} = 0.2x - 0.001xy$, $\frac{dy}{dt} = -0.4y + 0.0000016xy$.
(a) Which of the populations, foxes or rabbits, is described by x? Which one is described by y? Explain your answer.

(b) Find equilibrium solutions and explain their meaning.

(c) Find an expression for dy/dx and interpret it as a differential equation.

Continued on next page

4. The half-life of a radioactive substance is 12 years. Suppose we have a 1000 grams sample.
(a) Find the mass that remains after t years.

(b) Estimate the time needed for the substance to decay to 100 grams.

(c) Find the time needed for the substance to decay to 15 % of its original amount.

Continued on next page

5. [Context of question 11 on page 539] The half life of the carbon ${}^{14}C$ is 5730 years. An object was found, that contains 16 % as much ${}^{14}C$ radioactivity as the corresponding material on Earth today.
(a) Estimate the age of the object.

(b) Assume that the measurement of the ${}^{14}C$ radioactivity was off by 1 % (i.e., the object contains 15-17 % as much ${}^{14}C$ radioactivity as the corresponding material on Earth today). Give an estimate (in terms of an interval) of the age of the object.

Continued on next page

6. Solve the equation

$$\frac{dP}{dt} = 3P\left(1 - \frac{P}{10}\right), \qquad P(0) = 1.$$

Continued on next page

7. The population of long-nosed-short-eared amber-brown ants has been modeled by the differential equation

$$\frac{dA}{dt} = kA\left(1 - \frac{A}{K}\right),$$

where $A(t)$ is the biomass in kilograms at time t (biomass is the total mass of all members of population). The carrying capacity is estimated to be K=60000 kilograms, and k=0.66 per year.

(a) If $A(0) = 10000$, find the biomass two years later.

(b) If $A(0) = 10000$, find the biomass ten years later.

(c) How long will it take for the biomass to reach 59000 kilograms?

THE END

Name: ____________________

Arts Sci 1D06	Winter 2012	Assignment 15

Material covered: Sections 8.1, 8.2.

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1. (a) Find the limit of the sequence $a_n = 3\ln(4n+3) - \ln(n^3-1)$.

(b) Determine whether the sequences $b_n = \sin(n\pi/2)$ and $c_n = \cos(n\pi/2)$ are convergent or not. If they are convergent, find their limit.

Continued on next page

2. (a) True or false: if the sequence $\{a_n\}$ is convergent and the sequence $\{b_n\}$ is divergent then the sequence $\{a_n b_n\}$ is convergent.

(b) True or false: if $\lim_{n\to\infty} a_n = 0$, then $\sum_{i=1}^{\infty} a_n$ is convergent.

3. Find the limit of the sequence $a_n = \dfrac{4^n n!}{(n+2)!}$.

Continued on next page

4. It is known that $\lim_{n\to\infty}(0.6)^n = 0$ (why?). Find n such that $(0.6)^n < 10^{-10}$.

5. Determine whether the series $\sum_{n=0}^{\infty} \frac{6^{2n+1}}{3^{4n+1}}$ is convergent or not. If it is convergent, find its sum.

Continued on next page

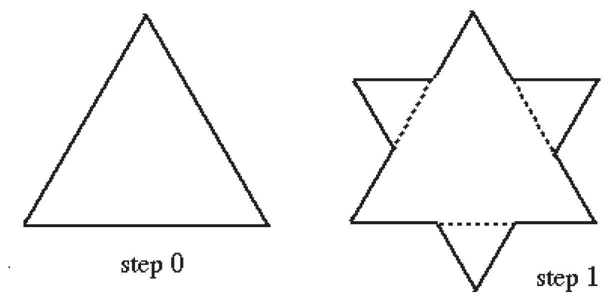
6. (a) Is the series $\sum_{n=10}^{\infty} \arctan\left(\frac{n^3}{n^3-n}\right)$ convergent or not?

(b) Determine whether the series $\sum_{n=0}^{\infty} \frac{\sqrt{n^3+1}}{(n^2+13)^2}$ is convergent or not.

(c) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{e^{3n}}$.

Continued on next page

7. Start with an equilateral triangle of side 1 (call that step 0). Divide each side into three equal parts and construct an equilateral triangle over the middle part (that is step 1). In step 2, repeat the above process by building an equilateral triangle over each of the 12 sides. If you keep repeating this process indefinitely, you will obtain the Koch snowflake curve.



(a) Find the number of sides of the polygon that is obtained in the n-th step. Find the length of each side and the total length of the curve.

Continued on next page

(b) Find the length of the snowflake curve.

(c) Find the area of the region bounded by the snowflake curve.

THE END

Name: ______________________

Arts Sci 1D06	Winter 2012	Assignment 16

Material covered: Sections 8.3, 8.4, 8.5, 8.6.

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1. Consider the series $\sum_{n=1}^{\infty} ne^{-2n}$.

(a) Check that all asumptions of the integral test are satisfied.

(b) Determine whether the given series is convergent or not.

Continued on next page

2. Determine whether the following series are convergent or not.

(a) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{3+e^n}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2-n-1}{3n^3+66n+1}$.

Continued on next page

3. Use the ratio test to determine whenter the following series converge or not.

(a) $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$.

(b) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$.

(c) $\sum_{n=0}^{\infty} \frac{4^n}{n\,3.99^n}$.

Continued on next page

4. True/false questions.

(a) If $\sum_{n=0}^{\infty} a_n 3^n$ is convergent, then $\sum_{n=0}^{\infty} a_n 4^n$ is convergent.

(b) If $\sum_{n=0}^{\infty} a_n 3^n$ is divergent, then $\sum_{n=0}^{\infty} a_n 4^n$ is divergent.

(c) If $\sum_{n=0}^{\infty} a_n 3^n$ is convergent, then $\sum_{n=0}^{\infty} a_n (-3)^n$ is convergent.

Continued on next page

5. (a) Find a power series representation of the function $f(x) = \dfrac{1}{3+4x}$.

(b) What is the radius of convergence of the series in (a)?

(c) Use (a) to find a power series representation of $\ln(3+4x)$.

(d) What is the radius of convergence of the series in (c)?

Continued on next page

6. Determine the radius of convergence for the following series

(a) $\sum_{n=0}^{\infty} \frac{x^n}{n\,14^n}$.

(b) $\sum_{n=0}^{\infty} n\,x^n$.

(c) $\sum_{n=0}^{\infty} n!\,x^n$.

Continued on next page

7. Determine the radius of convergence and the interval of convergence for the series

$$\sum_{n=0}^{\infty} \frac{3n}{5^n}(3x-1)^n.$$

Continued on next page

8. Use a power series to determine the value of the integral $\int_0^{0.1} \frac{1}{1+x^4}\, dx$ to four decimal places.

9. Evaluate $\int \arctan(x^3)\, dx$ as a power series.

THE END

Name: ______________________

Arts Sci 1D06 Winter 2012 Assignment 17

Material covered: Sections 8.6, 8.7, 8.8; few review questions about series.

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1. Find the Taylor series for the following functions.

(a) $f(x) = e^x$ centred at $x = 0$.

(b) $f(x) = e^x$ centred at $x = 1$.

(c) $f(x) = e^x$ centred at $x = -1$.

Continued on next page

2. (a) Write down the Maclaurin series expansion of e^x.

(b) Using (a), find the Maclaurin series expansion of xe^{x^2-2}.

(c) Write down the Maclaurin series expansions for $\sin x$ and $\cos x$.

(d) Using (c), find the Maclaurin series expansion of $\sin(x+1)$.

Continued on next page

3. (a) Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{2^{2n+1}(2n+1)!}$.

(b) Using series, find the limit $\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3 - \frac{1}{120}x^5}{x^7}$.

(c) Expand $(1 - x^3)^{1/3}$ as a power series and find its radius of convergence.

Continued on next page

4. (a) Evaluate the indefinite integral $\int \sin(x^3)\,dx$ as an infinite series.

(b) Evaluate the indefinite integral $\int e^{-x^2}\,dx$ as an infinite series.

Continued on next page

5. Express 5.4114114114114... as a fraction.

6. Is the series $\sum_{n=1}^{\infty} \frac{(n+4)!}{n!7^n}$ absolutely convergent?

Continued on next page

7. (a) Find an example of a series that is convergent but is not absolutely convergent.

(b) Is it true that if $\sum_{n=1}^{\infty} a_n$ is divergent, then the series $\sum_{n=1}^{\infty} |a_n|$ is also divergent?

8. How many terms of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n}$ do we have to add in order to find the sum up to an error of less than 0.001?

THE END

Name: ______________________

Arts Sci 1D06	Winter 2012	Assignment 18

Material covered: Sections 11.1, 11.2, 11.3.

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1. (a) Sketch the domain of the function $f(x, y) = (x^2 - y^2)^{-1/2}$.

(b) Sketch the domain and find the range of the function $f(x, y) = \arcsin(xy)$.

Continued on next page

2. Draw a contour map of the given function, showing (and labeling) several level curves.
(a) $f(x, y) = xy$

Continued on next page

(b) $f(x,y) = e^{1/(x^2+y^2)}$.

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3. Find the range of the function $f(x, y, z, t) = \dfrac{xy - ze^t}{x^2 + y^2}$.

4. Find the domain and sketch the graph of the function $f(x, y) = \sqrt{4 - x^2 - 2y^2}$.

Continued on next page

5. (a) Determine the largest set on which the function $z = \ln x \ln y$ is continuous.

(b) Determine the largest set on which the function $z = \ln(3x - y)$ is continuous.

(c) Find all points where the function $f(x, y) = \cos(x - y) + \sec(x - y)$ is not continuous.

Continued on next page

6. Let $f(x,y) = \int_{xy}^{2} e^t dt + \int_{x}^{x^2} t^2 dt$.

(a) Find $f(0,1)$.

(b) Write down the statement of the Fundamental Theorem of Calculus, Part I.

(c) Find $f_x(x,y)$ and $f_y(x,y)$.

Continued on next page

7. (a) Does the function $u(x,t) = \sin(x - 2t) + 3\ln(x + 2t)$ satisfy the wave equation $u_{tt} = 4u_{xx}$?

(b) The kinetic energy of a body of mass m and velocity v is $K = \frac{1}{2}mv^2$. Show that $\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} = K$.

(c) Use differentials to approximate $f(0.03, 2.94)$, where $f(x,y) = \sqrt{1 - x^2 + y}$.

Continued on next page

8. Let N_{Mac} be the number of people who consider buying a Macintosh computer and let N_{PC} be the number of people who consider buying a comparable PC. P_{Mac} and P_{PC} are the prices of a Macintosh and a PC respectively. Find the signs of

$$\frac{\partial N_{Mac}}{\partial P_{PC}} \qquad \text{and} \qquad \frac{\partial N_{PC}}{\partial P_{Mac}}.$$

THE END