McMaster University Arts and Science 1D06 2013
Winter Midterm
March 5 2013

Duration: 90 minutes

Instructor: Dr. D. Haskell

Name: Solutions to Post.

Student ID Number: ______________ **TA:** ________________

This test paper is printed on both sides of the page. There are 8 question on 8 pages. You are responsible for ensuring that your copy of this test is complete. Bring any discrepancies to the attention of the invigilator.

Instructions

(1) Only the standard McMaster calculator is allowed.
(2) All answers must be written in the space following the question. If you need more space, ask the invigilator for more paper and indicate clearly where to find the answer.

Problem	Points
1 [10]	
2 [5]	
3 [6]	
4 [6]	
5 [6]	
6 [5]	
7 [6]	
8 [6]	
Total [50]	

2) [5 points] Find the general solution of the linear differential equation $x^2\frac{dy}{dx} = 3xy + 5x^4e^x$.

Linear ODE $\frac{dy}{dx} = \frac{3}{x}y + 5x^2e^x$

$$\frac{dy}{dx} - \frac{3}{x}y = 5x^2e^x$$

$$I(x) = e^{\int -3/x \, dx}$$

$$= e^{-3\ln(x)}$$

$$= e^{\ln x^{-3}}$$

$$= x^{-3}$$

$$x^{-3}\frac{dy}{dx} - 3x^{-2}y = x^{-3}5x^2e^x$$

$$\frac{d}{dx}\left(x^{-3}y\right) = 5\frac{1}{x}e^x$$

$$\frac{1}{x^3}y = \int 5\frac{1}{x}e^x\,dx$$

This ~~step~~ function does not have an elementary antiderivative.

3) [6 points] Solve the initial value problem

$$\frac{dy}{dx} = \frac{y\sin^2(x)}{(y+1)\sec(x)}\ ; \quad y(0) = e\ .$$

You do not need to express y explicitly as a function of x.

Separate the variables: $\frac{y+1}{y}\,dy = \frac{\sin^2(x)}{\sec(x)}\,dx$

$$\int\left(1+\frac{1}{y}\right)dy = \int \sin^2(x)\cos(x)\,dx$$

substitute $u = \sin(x)$

$$y + \ln|y| = \frac{1}{3}\sin^3(x) + C$$

When $x=0$, $y=e$: $e + \ln(e) = \frac{1}{3}\sin^3(0) + C$

$$e + 1 = C$$

$$y + \ln(y) = \frac{1}{3}\sin^3(x) + e + 1.$$

4) [6 points] Newton's Law of Heating states that the rate of heating of an object is proportional to the difference between the temperature of the object and the temperature of the surroundings. Let A be the (constant) temperature of the surroundings and $T(t)$ be the temperature of the object at time t.

a) Write a differential equation for T.

$$\frac{dT}{dt} = k(A - T).$$

b) A cake at room temperature (20°C) is put into an oven at 215°C at time $t = 0$. After ten minutes, the cake has reached 50°C. How many minutes does it take for the temperature of the cake to reach 125°C?

$T(0) = 20$, $A = 215$

$T(10) = 50$, find t so $T(t) = 125$.

$$\frac{dT}{dt} = k(A-T)$$

$$\int \frac{1}{A-T}\,dT = \int k\,dt$$

$$-\ln|A-T| = kt + C$$

$$|A-T| = e^{-kt}e^{-C}$$

$$T = A - Be^{-kt},$$

where $B = \pm e^{-C}$.

~~For~~ $A = 215$

$$T(0) = 215 - Be^{-0} = 20$$

$$195 = B.$$

$$T(10) = 215 - 195e^{-k10} = 50$$

$$e^{-k10} = \frac{-165}{-195}$$

$$k = -\frac{1}{10}\ln\left(\frac{165}{195}\right)$$

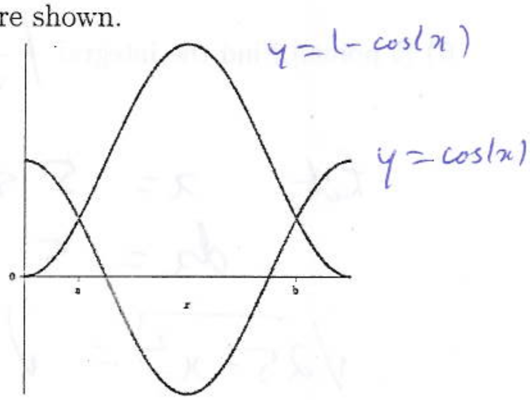
$$T(t) = 125 = 215 - 195e^{-kt}$$

$$\frac{90}{195} = e^{-kt}$$

$$t = \frac{10}{\ln\left(\frac{165}{195}\right)}\ln\left(\frac{90}{195}\right)$$

$$\underline{t = 46.28}$$

5) [6 points] The graphs of $y = \cos(x)$ and $y = 1 - \cos(x)$ are shown.

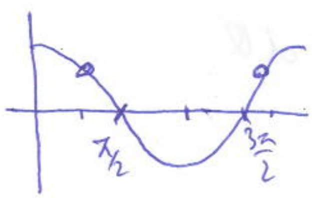


a) Find the x coordinates a and b of the points at which the curves intersect.

$$1 - \cos(x) = \cos(x)$$

$$1 = 2\cos(x)$$

$$\cos(x) = \frac{1}{2} \qquad x = \frac{\pi}{3}, \ \frac{5\pi}{3}.$$



b) Find the area of the region bounded by the curves for $a \le x \le b$.

$$\text{area} = \int_{\pi/3}^{5\pi/3} \left(1 - \cos(x) - \cos(x)\right) dx$$

$$= \int_{\pi/3}^{5\pi/3} \left(1 - 2\cos(x)\right) dx$$

$$= \Big[x - 2\sin(x) \Big]_{\pi/3}^{5\pi/3}$$

$$= \frac{5\pi}{3} - 2\sin\left(\frac{5\pi}{3}\right) - \frac{\pi}{3} + 2\sin\left(\frac{\pi}{3}\right)$$

$$= \frac{4\pi}{3} - \frac{2(-\sqrt{3})}{2} + 2\frac{\sqrt{3}}{2} = \frac{4\pi}{3} + 2\sqrt{3}$$

6) [5 points] Find the integral $\int \frac{1}{\sqrt{25-x^2}}\,dx$.

Let $x = 5\sin(\theta)$

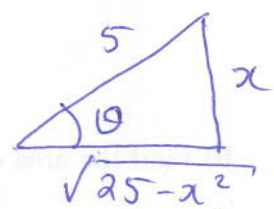
$dx = 5\cos(\theta)\,d\theta$

$$\sqrt{25-x^2} = \sqrt{25-25\,\cancel{\cos}\sin^2\theta} = \sqrt{25\,(1-\cancel{\cos}\sin^2\theta)}$$

$$= \sqrt{25\cos^2\theta} = 5\cos\theta$$

$$\int \frac{1}{\sqrt{25-x^2}}\,dx = \int \frac{1}{5\cancel{\cos\theta}} \cdot 5\cancel{\cos\theta}\,d\theta$$

$$= \theta + C$$

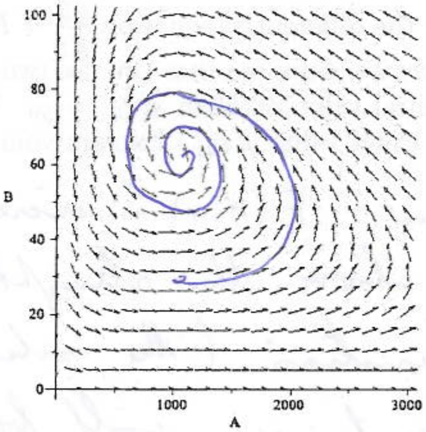


not needed

$$= \arcsin\left(\frac{x}{5}\right) + C.$$

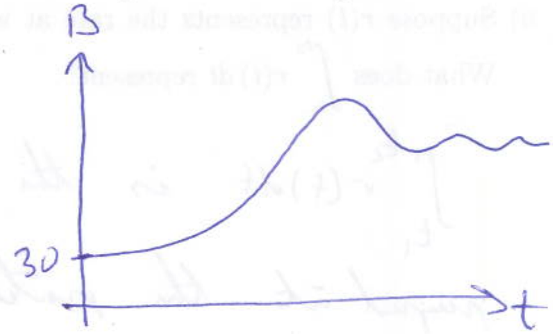
7) [6 points] The populations $A(t)$, $B(t)$ of two interacting species are modelled by the differential equations

$$\begin{aligned} \frac{dA}{dt} &= 8000A - 1.6A^2 - 100AB \, , \\ \frac{dB}{dt} &= -2000B + 2AB \, . \end{aligned}$$



a) On the slope field given, sketch the solution curve which starts at $A = 1000$, $B = 30$ (hint: the solution approaches an equilibrium solution).

b) For the solution curve that you drew, sketch the graphs of A against t and B against t.



c) Find the (non-zero) equilibrium solution of the system.

Solve $\frac{dA}{dt} = 0$ and $\frac{dB}{dt} = 0$.

$$\frac{dA}{dt} = 8000A - 1.6A^2 - 100AB = 0$$

$$A(8000 - 1.6A - 100B) = 0$$

$$A = 0 \text{ or } 8000 - 1.6A - 100B = 0.$$

$$\frac{dB}{dt} = -2000B + 2AB = 0$$

$$B(-2000 + 2A) = 0$$

$$B = 0 \text{ or } \underline{A = 1000}$$

Substitute $A = 1000$ in 1st eq^n:

$$8000 - 1600 - 100B = 0$$

$$6400 = 100B$$

$$\underline{B = 64}$$

8) [6 points]

a) Consider the differential equation $\frac{dy}{dx} = F(x, y)$, where F is a function which increases as x and y increase. Suppose that you use two steps of Euler's method to approximate a solution to the initial value problem $y(x_0) = y_0$. Will the approximation y_2 be greater than or less than the exact value $y(x_2)$? Explain your answer.

Because $F(x,y)$ is increasing, the solution curve will curve above the straight line which is used for the approximation (the solution is concave up). So the approximation y_2 will be less than the exact value $y(x_2)$.

b) Suppose $r(t)$ represents the rate at which an intravenous drug is pumped into a patient. What does $\int_{t_1}^{t_2} r(t)\,dt$ represent?

$\int_{t_1}^{t_2} r(t)\,dt$ is the amount of the drug which is pumped into the patient from time t_1 to time t_2.

c) Suppose the function $f(x)$ is continuous on $[0, \infty)$ and the improper integral $\int_1^\infty f(x)\,dx$ converges to a finite value. Must $\int_0^\infty f(x)\,dx$ also converge to a finite value? Explain your answer.

$$\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx.$$

$\int_0^1 f(x)\,dx$ is finite as f is continuous on $[0,\infty)$.

$\int_1^\infty f(x)\,dx$ is finite by hypothesis. Hence the sum is also finite, so the integral converges.