1) [10 points]

a) State the Integral Test for convergence of the series $\sum_{n=0}^{\infty} a_n$.

If $a_n = f(n)$ and the function $f(x)$ is positive and decreasing then the series and the improper integral $\int_1^\infty f(x)\,dx$ either both converge or both diverge.

b) State the Fundamental Theorem of Calculus Part I.

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

c) Find $\int_1^2 x^2\,dx$. $= \left[\frac{1}{3}x^3\right]_1^2 = \frac{1}{3}8 - \frac{1}{3}1 = \frac{7}{3}$

d) The volume V of a right circular cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$. Find the rate of change of V as r changes and as h changes.

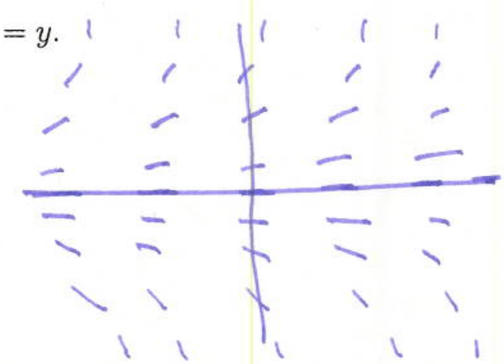
$$\frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2, \quad \frac{\partial V}{\partial r} = \frac{1}{3}\cdot 2\pi r h$$

e) Sketch the slope field for the differential equation $\frac{dy}{dx} = y$.

$y > 0$, $\frac{dy}{dx} > 0$

$y = 0$, $\frac{dy}{dx} = 0$

$y < 0$, $\frac{dy}{dx} < 0$



3) [8 points]

a) State the domain of f if $f(x,y) = \dfrac{\sqrt{4-x^2}}{y^2+3}$.

$y^2+3 \neq 0$. $\sqrt{4-x^2}$ is defined if $4-x^2 \geq 0$ i.e. $4 \geq x^2$

$-2 \leq x \leq 2$

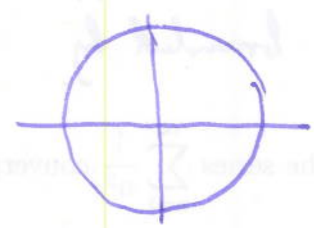
Domain is $\{(x,y) : -2 \leq x \leq 2,\ y \text{ is anything}\}$.

b) Sketch the level curve $k = 4$ for $f(x,y) = \sqrt{x^2+y^2}$.

$\sqrt{x^2+y^2} = 4$

$x^2+y^2 = 16$

circle centre $(0,0)$ radius 4



c) Does the following limit exist? Justify your answer.

$$\lim_{(x,y)\to(0,0)} \frac{-xy}{x^2+y^2}$$

$x=0$ $\quad \dfrac{-xy}{x^2+y^2} = \dfrac{0}{y^2} = 0$ for $y \neq 0$

$y=0$ $\quad \dfrac{-xy}{x^2+y^2} = \dfrac{0}{x^2} = 0$ for $x \neq 0$.

$x=y$ $\quad \lim_{(x,x)\to(0,0)} \dfrac{-xy}{x^2+y^2} = \lim_{x\to 0} \dfrac{-x^2}{x^2+x^2} = -\dfrac{1}{2}$

$x=-y$ $\quad \lim_{(x,y)\to(0,0)} \dfrac{-xy}{x^2+y^2} = \lim_{x\to 0} \dfrac{x^2}{x^2+x^2} = +\dfrac{1}{2}$

Since $\lim_{(x,y)\to(0,0)} \dfrac{-xy}{x^2+y^2}$ can take different values depending on the path to $(0,0)$, the limit does not exist.

5) [8 points] Find the indefinite integral $\int \frac{3x-1}{x^2-3x-10}\,dx$.

$$\frac{3x-1}{x^2-3x-10} = \frac{3x-1}{(x-5)(x+2)} = \frac{A}{x-5} + \frac{B}{x+2}$$

$$3x-1 = A(x+2) + B(x-5).$$

substitute $x=-2$: $\quad -6-1 = A(-2+2) + B(-2-5)$

$$-7 = B(-7) \qquad \underline{B=1}$$

substitute $x=5$: $\quad 15-1 = A(5+2) + B(5-5)$

$$14 = 7A \qquad \underline{A=2}$$

$$\int \frac{3x-1}{x^2-3x-10}\,dx = \int\left(\frac{2}{x-5} + \frac{1}{x+2}\right)dx$$

$$= 2\ln|x-5| + 1\ln|x+2| + C.$$

7) [8 points] Test the series $\sum_{n=2}^{\infty} \frac{8^n}{2+9^n}$ for convergence. State any test you use, and be careful to check its hypotheses.

$2+9^n \geq 9^n$

so $\frac{1}{2+9^n} \leq \frac{1}{9^n}$

$\frac{8^n}{2+9^n} \leq \frac{8^n}{9^n}$

thus the series is bounded above by $\sum \frac{8^n}{9^n}$.

This is a geometric series with ratio $\frac{8}{9} < 1$, hence this series converges

So $\sum \frac{8^n}{2+9^n}$ converges also, by the comparison test.

10) [8 points] Find the improper integral $\int_1^\infty (1 - \frac{1}{x})e^{\ln(x)-x}\, dx$.

$u = \ln(x) - x$

$du = (\frac{1}{x} - 1)dx$

$= -(1 - \frac{1}{x})dx$

$$\int \left(1 - \frac{1}{x}\right) e^{\ln(x)-x}\, dx = \int e^u (-du) = -e^u + C = -e^{\ln(x)-x} + C.$$

$$\int_1^\infty \left(1 - \frac{1}{x}\right) e^{\ln(x)-x}\, dx = \lim_{t\to\infty} \left[-e^{\ln(x)-x}\right]_1^t$$

$$= \lim_{t\to\infty} \left(-e^{\ln(t)-t} + e^{\ln(1)-1}\right)$$

$$\lim_{t\to\infty} (\ln(t) - t) = \lim_{t\to\infty} \ln(t)\left(1 - \frac{t}{\ln(t)}\right)$$

$$= \infty \cdot (-\infty) = -\infty$$

Find $\lim_{t\to\infty} \frac{t}{\ln(t)}$ with L'Hospital's Rule

So $\lim_{t\to\infty} \left(-e^{\ln(t)-t} + e^{-1}\right)$

$$= -e^{\lim_{t\to\infty}(\ln(t)-t)} + e^{-1} = -e^{-\infty} + e^{-1}$$

$$= e^{-1}.$$

Thus $\int_1^\infty \left(1 - \frac{1}{x}\right) e^{\ln(x)-x}\, dx = \frac{1}{e}.$

12) [12 points]

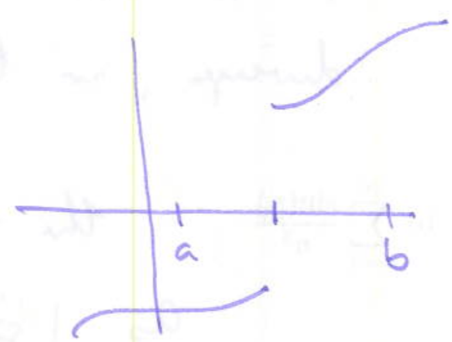
a) What can you say about solutions to the differential equation $\frac{dy}{dx} = x^2 + y^2$ just by looking at the differential equation?

As $x^2+y^2 \geq 0$ for all x, y, $\frac{dy}{dx} \geq 0$ so y is an increasing function of x.

b) Sketch the graph of a function that illustrates how the hypotheses are needed in the Intermediate Value Theorem.

IVT If f is continuous and $f(a) < L < f(b)$ then there is $c \in (a,b)$ with $f(c) = L$.

Continuity is needed: for the given function, there is no $c \in (a,b)$ with $f(c) = 0$.



c) Suppose $\sum a_n$ is a convergent sum of positive terms. Does it follow that $\sum (-1)^n a_n$ converges?

If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

By the alternating series test, $\sum (-1)^n a_n$ also converges.