ASID06 practice problems for midyear exam

#12 For what values of c is the function

$$g(x) = cx + \frac{1}{x^2+3} \quad \text{increasing on } (-\infty, \infty)?$$

Solution: As a function is increasing if and only if its derivative is positive, the question is asking: for what values of c is it true that $g'(x) > 0$ for all x.

$$g'(x) = c + (-1)(x^2+3)^{-2}\, 2x$$

So $g'(x) > 0$ provided $c > \dfrac{2x}{(x^2+3)^2}$, for all x.

Now we need to know the largest value that the function $h(x) = \dfrac{2x}{(x^2+3)^2}$ takes on (if there is such a largest value).

$h(x)$ is ~~maximum~~ possibly maximised when $h'(x) = 0$.

$$h'(x) = \frac{(x^2+3)^2\, 2 - 2x \cdot 2(x^2+3)^1 \cdot 2x}{(x^2+3)^4}$$

$$= \frac{2(x^2+3) - 8x^2}{(x^2+3)^3}$$

$h'(x) = 0$ when $\quad 2(x^2+3) - 8x^2 = 0$

$$-6x^2+6 = 0$$

$$x^2 = 1$$

$$x = \pm 1.$$

	$\frac{1}{(x^2+3)^3}$	$-6x^2+6$	f'	f	
$(-\infty, -1)$	+	−	−	dec	
					min
$(-1, 1)$	+	+	+	inc	
					max
$(1, \infty)$	+	−	+	dec	

$$h(1) = \frac{2}{(1+3)^2} = \frac{1}{8} \text{ is a local max.}$$

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{2x}{(x^2+3)^2} = 0$$ (by L'H, or because denominator degree is larger than numerator degree)

and $\lim_{x\to-\infty} h(x) = 0$

So the value $h(1) = \frac{1}{8}$ is an absolute maximum as well as a local maximum.

Thus $g'(x) \geq 0$ for all x provided $c \geq \frac{1}{8}$,

hence g is increasing for all x if $c \geq \frac{1}{8}$.