Arts & Science 1D06 Quiz #7 14 January 2016

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Please provide detailed solutions to the problems below. Correct responses without justification may not receive full credit. The use of a calculator is permitted.

[6 marks] (1) Consider the sequence

$$\left\{0, -\frac{3}{4}, \frac{8}{8}, -\frac{15}{16}, \frac{24}{32}, -\frac{35}{64}, \dots\right\}$$

(a) *[3]* Find an expression for the general term in the sequence, a_n.

$$a_n = (-1)^{n+1}\frac{n^2-1}{2^n}$$

(b) *[3]* Does the sequence converge? If so, to what value does it converge?

Consider the sequence with terms $|a_n| = \dfrac{n^2-1}{2^n}$. If $|a_n| \to 0$, then we also know that $a_n \to 0$. We'll want to use L'Hôpital's Rule to show $|a_n| \to 0$, so we'll switch to a continuous function: $f(x) = \dfrac{x^2-1}{2^x}$.

$$\lim_{x\to\infty}\frac{x^2-1}{2^x} \overset{\text{LH}}{=} \lim_{x\to\infty}\frac{2x}{2^x\ln 2} \overset{\text{LH}}{=} \lim_{x\to\infty}\frac{2}{2^x\ln^2 2} = 0$$

Thus we have $|a_n| \to 0$ and $a_n \to 0$ as well.

[4 marks] (2) Show that the following sequence is monotone decreasing. Does the sequence converge?

$$\{ne^{-n}\}_{n=1}^{\infty}$$

To show that the sequence is monotone decreasing, we'll consider $g(x) = xe^{-x}$.

$$g'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) < 0 \text{ when } x > 1.$$

So we know that $g(x)$ is decreasing on $(1, \infty)$, which implies that our sequence is monotone decreasing. Since $\{ne^{-n}\}_{n=1}^{\infty}$ is a monotone decreasing sequence that is bounded (above by e^{-1} and below by 0), by the monotone convergence theorem, $\{ne^{-n}\}_{n=1}^{\infty}$ converges.