Arts & Science 1D06 Quiz #8 SOLUTIONS 29 January 2015

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TA:______________________________

Please provide detailed solutions to the problems below. Correct responses without justification may not receive full credit. The use of a calculator is permitted.

[5 marks] (1) Use the comparison test to prove that this series diverges:

$$\sum_{n=1}^{\infty} \frac{(4+\sin(n))^n}{2^n}.$$

The goal is to show that the series in question is larger than some other divergent series. Well, we know that $\sin(n) > -1$ for any n. This means that:

$$\sum_{n=1}^{\infty} \frac{(4+\sin(n))^n}{2^n} > \sum_{n=1}^{\infty} \frac{(4-1)^n}{2^n} = \sum_{n=1}^{\infty} \frac{3^n}{2^n}.$$

The latter is a geometric series with $r = \frac{3}{2} > 1$, so it diverges. Thus, by the comparison test, $\frac{(4+\sin(n))^n}{2^n}$ diverges too.

[5 marks] (2) Use the limit comparison test to show that this series converges.

$$\sum_{n=1}^{\infty} \frac{2^n + \ln(n)}{3^n}.$$

As $n \to \infty$, this series most resembles $\sum \frac{2^n}{3^n}$, which is a convergent geometric series. Taking the limit of the ratio of these two series:

$$\lim_{n\to\infty} \frac{2^n + \ln(n)}{3^n} \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{2^n + \ln(n)}{2^n} = \lim_{n\to\infty} 1 + \frac{\ln(n)}{2^n}.$$

Let's consider the real-valued limit $\lim_{x\to\infty} \frac{\ln(x)}{2^x}$ and use l'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln(x)}{2^x} \overset{\text{L'Hôp}}{\overset{\downarrow}{=}} = \lim_{x\to\infty} \frac{1}{x\ln(2)2^x} = 0.$$

So, as $n \to \infty$, the ratio between $\frac{2^n+\ln(n)}{3^n}$ and $\frac{2^n}{3^n}$ is 1. But the latter are the terms of a convergent series, which means by the limit comparison test that $\sum_{n=0}^{\infty} \frac{2^n + \ln(n)}{3^n}$ converges.