Name: ____________________

Student number: ____________________

ARTSCI 1D06

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DAY CLASS
DURATION OF EXAMINATION 2.5 Hours
MCMASTER UNIVERSITY FINAL EXAMINATION — PRACTICE VERSION

THIS EXAMINATION PAPER INCLUDES n QUESTIONS ON m PAGES. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

Special instructions: Answer all the questions in the space provided.
If you need more paper, ask the invigilator.
Use of Casio-FX-991 calculator only is permitted.
This paper must be returned with your answers.

1)

a) State the Mean Value Theorem.

b) State the limit comparison test for convergence of the series $\sum_{n=1}^{\infty} a_n$.

c) Let P be the point with cartesian coordinates $(-3,-4)$. Find polar coordinates for P.

d) Solve the separable differential equation $\frac{dy}{dx} = y^2x^2$, $y(0) = 1$.

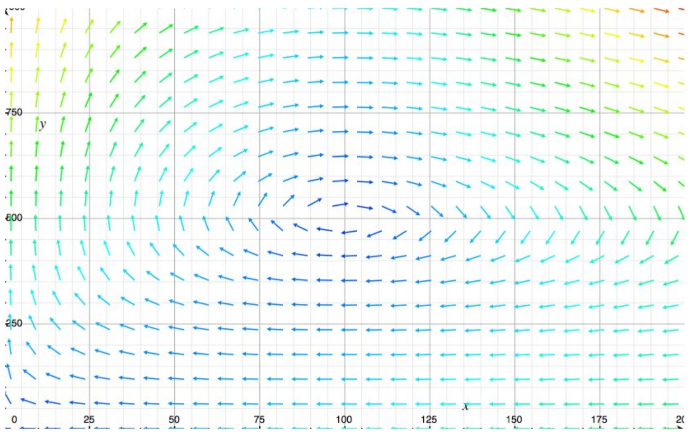
e) State the definition of the integral of the function $f(x)$ on the interval $[a, b]$ as the limit of Riemann sums.

f) Sketch the contour map for the function $z = x^2 - 9y^2$ (you should indicate at least 3 level sets).

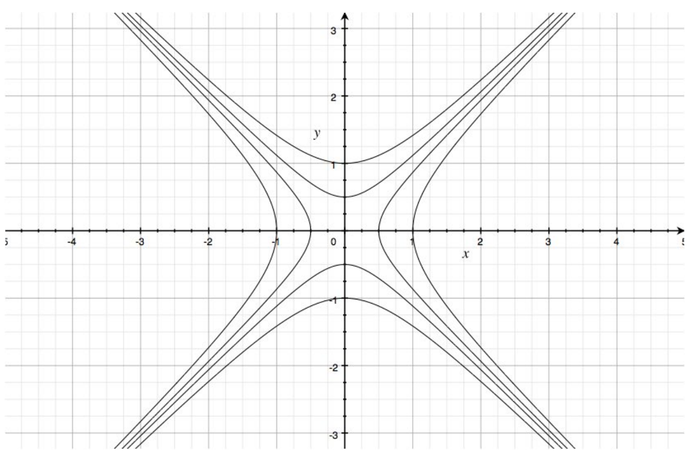
g) Sketch the curve given in polar form by the equation $r = 4$.

h) Find x_3 when Newton's method is used to approximate a zero of the function $f(x) = x - \cos(x)$ with starting point $x_1 = \pi/4$.

i) The direction field for the system of differential equations $\frac{dx}{dt} = -500x + xy$, $\frac{dy}{dt} = 200y - 2xy$ is given. Sketch the solution curve starting at the point $x = 100, y = 250$.



j) A contour map for the surface $z = f(x, y)$ is given. Find the approximate coordinates of the points where $f_x = 0$ and the points where $f_y = 0$.



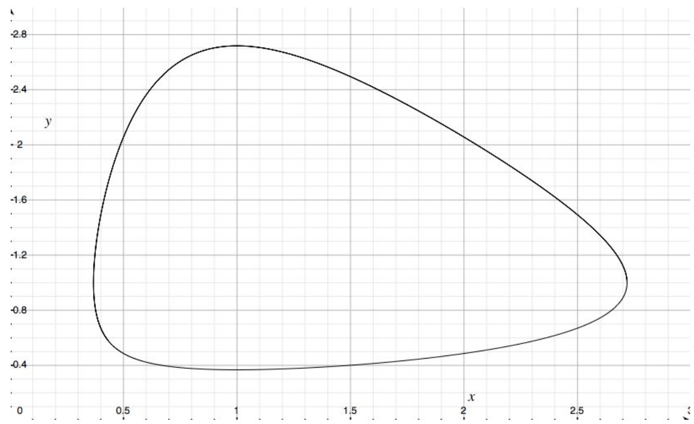
2) Let $f(x,y) = \dfrac{2xy}{x^2+y^2}$.

a) Show that this function is not continuous at the origin.

b) Find f_x and f_y, for $(x,y) \neq (0,0)$.

c) Find all second order partial derivatives and verify that $f_{xy} = f_{yx}$.

3) The graph of the curve parametrized by $x = e^{\cos\theta}$, $y = e^{\sin\theta}$ is shown. Find the exact value of the coordinates where the tangent line to the curve is horizontal, and the exact coordinates of the point where the tangent line is vertical.



4) Solve the initial-value problem $y' + 4xy = x, \quad y(0) = 1$.

5)

a) Find the Taylor series for the function $f(x) = \ln(1 + 2x)$ (do not quote a known Taylor series).

b) Find the radius of convergence for the above series.

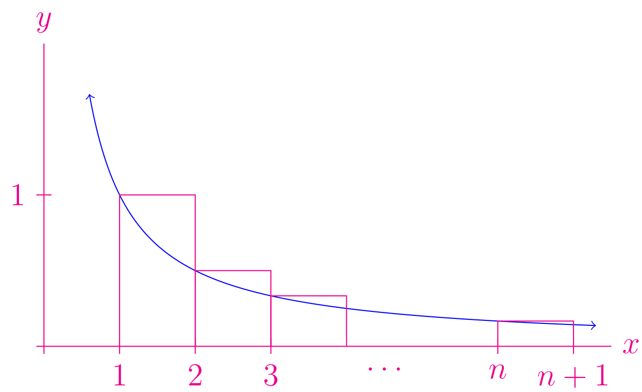
c) Find the interval of convergence for the series.

6) The goal of this problem is to justify the integral test for convergence of a series. Let $\{a_i\}$ be a sequence of positive terms and assume that f is a continuous, non-negative decreasing function with $a_i = f(i)$ for all i.

a) Write L_n for the Riemann sum with left endpoints and $\Delta x = 1$ which approximates the integral

$$\int_1^{n+1} f(x)\, dx,$$

where n is any integer greater than 1. (Here is the picture.)



Express L_n as a finite sum.

b) Compare the series $\sum_{i=1}^{n} a_i$ with the integral $\int_1^{n+1} f(x)\, dx$ to find a lower bound for $\sum_{i=1}^{n} a_i$.

c) Write R_n for the Riemann sum with right endpoints and $\Delta x = 1$ which approximates the integral

$$\int_1^{n+1} f(x)\,dx,$$

where n is any integer greater than 1. Express R_n as a sum.

d) Compare the series with the integral to find an upper bound for $\sum_{i=1}^{n} a_i$.

e) Use these bounds on $\sum_{i=1}^{n} a_i$ to deduce the statement of the integral test.

Formula Sheet

Integrals (constants of integration are omitted)

$$\int x^n\,dx = \frac{x^{n+1}}{n+1},\ n \neq -1 \qquad \int \frac{1}{x}\,dx = \ln|x|$$

$$\int e^x\,dx = e^x \qquad \int a^x\,dx = \frac{a^x}{\ln a}$$

$$\int \sin x\,dx = -\cos x \qquad \int \cos x\,dx = \sin x$$

$$\int \tan x\,dx = -\ln|\cos x| \qquad \int \cot x\,dx = \ln|\sin x|$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| \qquad \int \csc x\,dx = -\ln|\csc x + \cot x|$$

$$\int \sec^2 x\,dx = \tan x \qquad \int \csc^2 x\,dx = -\cot x$$

$$\int \sec x \tan x\,dx = \sec x \qquad \int \csc x \cot x\,dx = -\csc x$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)$$

Trigonometry

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

$$\sin(2x) = 2\sin x\cos x \quad \cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

Newton's method $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Euler's method $x_{n+1} = x_n + h,\ y_{n+1} = y_n + F(x_n, y_n)h$

Taylor series $T(x) = \sum_{n=0}^{\infty} f^{(n)}(a)(x-a)^n$