ArtSci 1D06 Calculus
Full year 2015–2016
Instructor: D. Haskell

Fall Midterm
Monday 26 October 2015 18:45–20:15

Instructions There are six questions on five pages. Answer all the questions in the space provided. If you need more paper, ask the invigilator.

NAME: Solutions

ID NUMBER:

TUTORIAL DAY AND TIME

Problem	Points
1 [10]	
2 [5]	
3 [5]	
4 [5]	
5 [5]	
6 [10]	
Total [40]	

1) [10 points]

a) Find $f'(x)$ for the function $f(x) = e^{x^2}$.

$$f'(x) = e^{x^2} 2x$$

b) Find $f'(x)$ for the function $f(x) = \dfrac{\cosh(x)}{x^3+2}$.

$$f'(x) = \frac{(x^3+2)\sinh(x) - (3x^2)\cosh(x)}{(x^3+2)^2}$$

c) Find $\dfrac{dy}{dx}$, where $x^2 + y^2 = 2xy$.

$$2x + 2y\frac{dy}{dx} = 2x\frac{dy}{dx} + 2y$$

$$(2y-2x)\frac{dy}{dx} = 2y - 2x$$

$$\frac{dy}{dx} = 1$$

d) Find $f'(x)$ for the function $f(x) = x^2 \cos(x)\sin(x)$.

$$f'(x) = 2x\cos(x)\sin(x) - x^2\sin^2(x) + x^2\cos^2(x)$$

e) Given $h(x) = \sin(f(x))$, $f(0) = \pi$ and $f'(0) = 2$, find $h'(0)$.

$$h'(x) = \cos(f(x))\, f'(x)$$

$$h'(0) = \cos(f(0))\, f'(0)$$

$$= \cos(\pi)\cdot 2$$

$$= -2.$$

2) [5 points] Consider the function

$$f(x) = \begin{cases} x^2 - 2x + 5, & \text{if } x \le 2; \\ -2x + 9, & \text{if } x > 2. \end{cases}$$

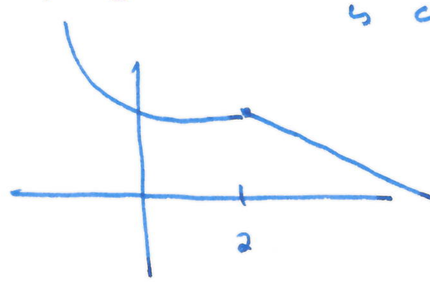
a) Find $\lim_{x \to 2^-} f(x)$, $\lim_{x \to 2^+} f(x)$. Is f continuous at $x = 2$? Justify your answer.

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 - 2x + 5) = 5$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (-2x + 9) = 5$

As the one-sided limits are equal, the function is continuous at $x = 2$.

b) Sketch the graph of $y = f(x)$.



3) [5 points] Let f be a function which is continuous and differentiable on a domain containing the interval (a, b). Suppose that $f(a) = a$ and $f(b) = b$ (such values are called *fixed points* of the function).

a) Show that there is some $c \in (a, b)$ with $f'(c) = 1$.

By the Mean Value theorem, there is $c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{b - a}{b - a}$$

$$f'(c) = 1.$$

b) Let $a = 0$ and $b = 1$. Give an example (other than the straight line) of a function which has this behaviour.

$f(x) = x^2$ has $f(0) = 0$, $f(1) = 1$.

So does x^n for any $n > 0$.

Or $f(x) = \sin\left(\frac{\pi}{2} \cdot x\right)$

4) [5 points] Find $f'(1)$ from the definition of the derivative for the function

$$f(x) = \begin{cases} -2x+4, & \text{if } x \le 1; \\ \dfrac{2}{x}, & \text{if } x > 1. \end{cases}$$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

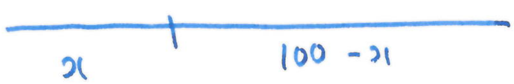
For $h>0$, $$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{\frac{2}{1+h} - (-2\cdot 1+4)}{h}$$

$$= \lim_{h\to 0^+} \left(\frac{2}{1+h} - 2\right)\frac{1}{h} = \lim_{h\to 0^+} \frac{-2h}{h} = -2.$$

For $h<0$, $$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{-2(1+h)+4-2}{h} = -2.$$

As $\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h}$, $f'(1) = -2$.

5) [5 points] A piece of wire 100 cm long is cut into two pieces (either of which could be of length 0). The first piece, of length x, is fashioned into an equilateral triangle. The second piece is fashioned into a square. Express the total area enclosed by the two pieces as a function of x. Find the maximum and minimum values of the total area. (The area of an equilateral triangle of side length ℓ is $\frac{\sqrt{3}}{4}\ell^2$).



area $\Delta = \frac{\sqrt{3}}{4}\left(\frac{x}{3}\right)^2$

area $\Box = \left(\frac{100-x}{4}\right)^2$.

total area $A = \frac{\sqrt{3}}{36}x^2 + \frac{1}{16}(100-x)^2$, $0 \le x \le 100$

$A' = \frac{\sqrt{3}}{18}x + \frac{1}{8}(100-x)(-1)$

$A' = 0$ when $x = 56.5$

$A(0) = 625$ max area is 625 (just square)

$A(56.5) = 272$ min area is 272

$A(100) = 481$

6) [10 points] Let $f(x) = \dfrac{1}{x} + \ln(x)$, with domain $(0, \infty)$.

a) Write $f(x) = \dfrac{1 + x\ln(x)}{x}$ to find $\lim_{x\to\infty} f(x)$ using L'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{1+x\ln(x)}{x} = \lim_{x\to\infty} \frac{\ln(x) + x\cdot\frac{1}{x}}{1} \quad \text{using L'H rule}$$

$$= \infty$$

b) Find $\lim_{x\to 0} x\ln(x)$ using L'Hôpital's Rule, and hence find $\lim_{x\to 0} f(x)$.

$$\lim_{x\to 0} x\ln(x) = \lim_{x\to 0} \frac{\ln(x)}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} \quad \text{by L'H}$$

$$= \lim_{x\to 0} -x = 0$$

$$\text{So } \lim_{x\to 0} \frac{1+x\ln(x)}{x} = \frac{1+0}{0} = \infty.$$

c) Find and classify all critical numbers of f.

$$f'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{1}{x^2}(-1+x).$$

$f' = 0$ when $x = 1$.

	$\frac{x-1}{x^2}$	f
$(0,1)$	$-$	inc
$(1,\infty)$	$+$	dec

so $f(1)$ is a local min

d) Sketch the graph of $y = f(x)$.