Name:________________
Student number:________________

ARTSCI 1D06

DEIRDRE HASKELL

DAY CLASS
DURATION OF EXAMINATION 2.5 Hours
MCMASTER UNIVERSITY FINAL EXAMINATION – PRACTICE VERSION

THIS EXAMINATION PAPER INCLUDES n QUESTIONS ON m PAGES. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

Special instructions: Answer all the questions in the space provided.
If you need more paper, ask the invigilator.
Use of Casio-FX-991 calculator only is permitted.
This paper must be returned with your answers.

Solutions

1)

a) State the Mean Value Theorem.

Let f be a function which is continuous on $[a,b]$ and differentiable on (a,b). Then there is some $c \in (a,b)$ with

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

b) State the limit comparison test for convergence of the series $\sum_{n=1}^{\infty} a_n$.

Assume $a_n \geq 0$ for all n. Let b_n be another sequence with $b_n > 0$ for all n. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite, non-zero, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

c) Let P be the point with cartesian coordinates $(-3,-4)$. Find polar coordinates for P.

$$r = \sqrt{x^2+y^2} = \sqrt{9+16} = \sqrt{25} = 5.$$

$$\tan\theta = \frac{y}{x} = \frac{-4}{-3}$$

$$\arctan\left(\frac{4}{3}\right) = 0.927 \text{ (radians)}$$

But P is in third quadrant, so angle should be

$$\pi + 0.927.$$

d) Solve the separable differential equation $\frac{dy}{dx} = y^2x^2$, $y(0) = 1$.

$$\int \frac{dy}{y^2} = \int x^2 \, dx$$

$$-y^{-1} = \frac{1}{3}x^3 + C$$

$$y = 1 \Big/ \left(-\frac{1}{3}x^3 - C\right)$$

$$y(0) = 1 = \frac{1}{-\frac{1}{3}\cdot 0a - C} = -\frac{1}{C}$$

$$C = -1.$$

$$y = \frac{1}{-\frac{1}{3}x^3 + 1}$$

e) State the definition of the integral of the function $f(x)$ on the interval $[a, b]$ as the limit of Riemann sums.

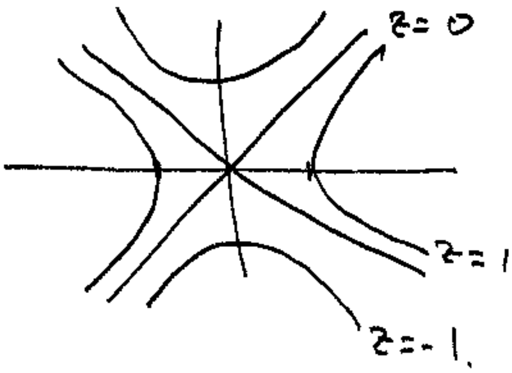
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$ where $\Delta x = \frac{b-a}{n}$ and x_i^* is a point in the ith interval of width Δx from a to b

f) Sketch the contour map for the function $z = x^2 - 9y^2$ (you should indicate at least 3 level sets).

$x^2 - 9y^2 = 0 \Rightarrow x = \pm 3y$

$x^2 - 9y^2 = 1$ goes through $(\pm 1, 0)$

$x^2 - 9y^2 = -1$ goes through $(0, \pm \frac{1}{3})$

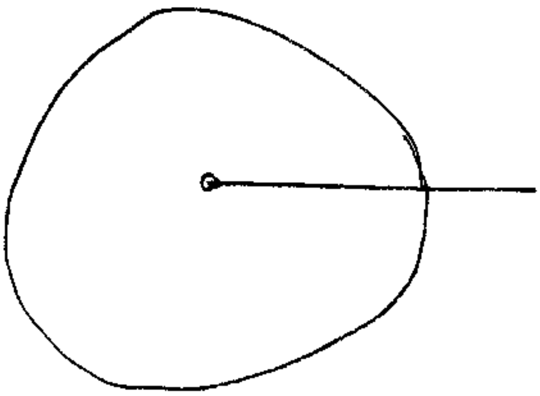


g) Sketch the curve given in polar form by the equation $r = 4$.

$r = 4$ radius constant,

θ any value

gives a circle, radius 4



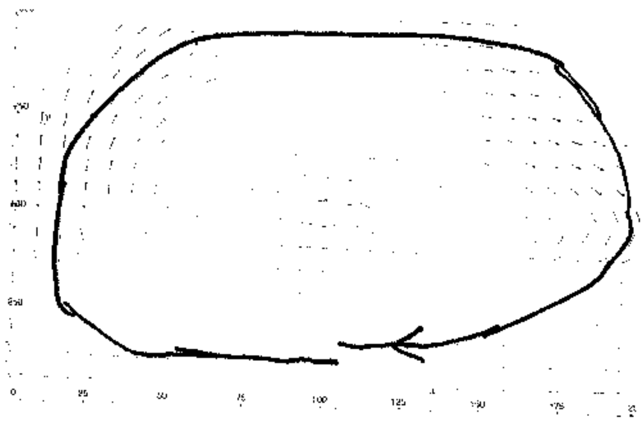
h) Find x_3 when Newton's method is used to approximate a zero of the function $f(x) = x - \cos(x)$ with starting point $x_1 = \pi/4$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad f'(x) = 1 + \sin(x)$$

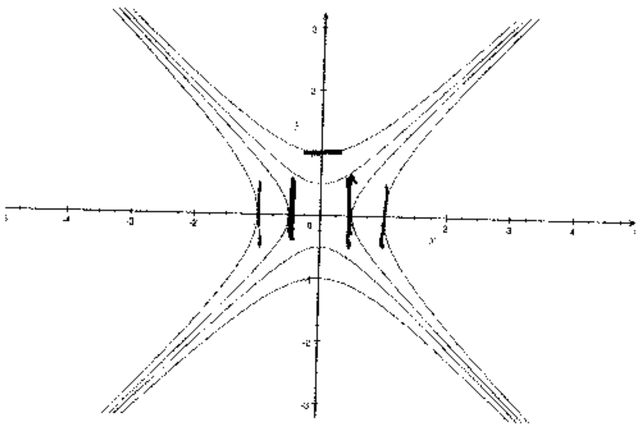
$$x_1 = \frac{\pi}{4}, \quad x_2 = \frac{\pi}{4} - \frac{\frac{\pi}{4} - \cos(\frac{\pi}{4})}{1 + \sin(\frac{\pi}{4})} = 0.7395$$

$$x_3 = 0.7395 - \frac{0.7395 - \cos(0.7395)}{1 + \sin(0.7395)} = 0.7391$$

i) The direction field for the system of differential equations $\frac{dx}{dt} = -500x + xy$, $\frac{dy}{dt} = 200y - 2xy$ is given. Sketch the solution curve starting at the point $x = 100, y = 250$.



j) A contour map for the surface $z = f(x, y)$ is given. Find the approximate coordinates of the points where $f_x = 0$ and the points where $f_y = 0$.



$f_x = 0$ when $y = 0$, $x =$ anything

$f_y = 0$ when $x = 0$, $y =$ anything.

2) Let $f(x,y) = \dfrac{2xy}{x^2+y^2}$.

a) Show that this function is not continuous at the origin.

Fix $x=0$ $\quad \lim_{(0,y)\to(0,0)} f(x,y) = \lim_{y\to 0} \frac{0}{0+y^2} = 0.$

Fix $x=y$ $\quad \lim_{(x,x)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{2x^2}{x^2+x^2} = \lim_{x\to 0} 1 = 1.$

as the two limit are not equal, the limit at $(0,0)$ does not exist, so function is not continuous at $(0,0)$.

b) Find f_x and f_y, for $(x,y) \neq (0,0)$.

$$f_x = \frac{(x^2+y^2)2y - 2xy(2x)}{(x^2+y^2)^2} = \frac{-2x^2y + 2y^3}{(x^2+y^2)^2}$$

$$f_y = \frac{(x^2+y^2)2x - 2xy(2y)}{(x^2+y^2)^2} = \frac{2x^3 - 2xy^2}{(x^2+y^2)^2}$$

c) Find all second order partial derivatives and verify that $f_{xy} = f_{yx}$.

$$f_{xx} = \frac{(x^2+y^2)^2(-4xy) - (-2x^2y+2y^3)2(x^2+y^2)2x}{(x^2+y^2)^4}$$

$$f_{xy} = \frac{(x^2+y^2)^2(-2x^2+6y^2) - (-2x^2y+2y^3)2(x^2+y^2)2y}{(x^2+y^2)^4}$$

$$f_{yy} = \frac{(x^2+y^2)^2(-4xy) - (2x^3-2xy^2)2(x^2+y^2)2y}{(x^2+y^2)^4}$$

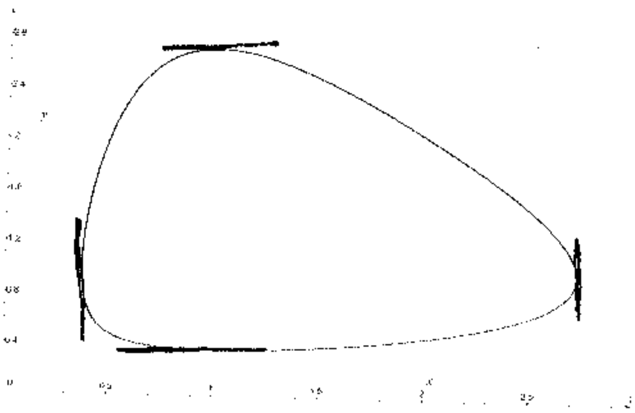
$$f_{yx} = \frac{(x^2+y^2)^2(6x^2-2y^2) - (2x^3-2xy^2)2(x^2+y^2)2x}{(x^2+y^2)^4}$$

Some algebra shows $f_{xy} = \dfrac{(x^2+y^2)(-2x^4-2y^4+12x^2y^2)}{(x^2+y^2)^4} = f_{yx}$

3) The graph of the curve parametrized by $x = e^{\cos\theta}$, $y = e^{\sin\theta}$ is shown. Find the exact value of the coordinates where the tangent line to the curve is horizontal, and the exact coordinates of the point where the tangent line is vertical.

$$\frac{dy}{d\theta} = \frac{dy}{dx} \cdot \frac{dx}{d\theta}, \text{ so}$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$



Horizontal tgt. line when

$\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$.

Vertical tgt line when $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$.

$$\frac{dy}{d\theta} = e^{\sin\theta} \cdot \cos\theta = 0 \quad \text{when} \quad \cos\theta = 0$$

$$\theta = \frac{\pi}{2} + n\pi.$$

$$\frac{dx}{d\theta} = e^{\cos\theta} \cdot (-\sin\theta) = 0 \quad \text{when} \quad \sin\theta = 0$$

$$\theta = n\pi.$$

~~Horizontal~~ Horizontal tgt line when $\theta = \frac{\pi}{2} + n\pi$, so $x = e^0 = 1$

$y = e^{\sin(\frac{\pi}{2})}$ or $y = e^{\sin(\frac{3\pi}{2})}$

$y = e^1$ or $y = e^{-1}$.

Vertical tgt line when $\theta = n\pi$, so $y = e^0 = 1$.

$x = e^{\cos(0)}$ or $x = e^{\cos(\pi)}$

$x = e^1$ or $x = e^{-1}$

4) Solve the initial-value problem $y' + 4xy = x$, $\quad y(0) = 1$.

Linear first-order, solve with integrating factor.

$$I(x) = e^{\int 4x\,dx} = e^{2x^2}$$

$$e^{2x^2} y' + e^{2x^2} \cdot 4x\, y = x e^{2x^2}$$

$$\frac{d}{dx}\left(e^{2x^2} y\right) = x e^{2x^2}$$

so $$\int \frac{d}{dx}\left(e^{2x^2} y\right) dx = \int x e^{2x^2}\, dx$$

$$e^{2x^2} y = \frac{1}{4} e^{2x^2} + C$$

$$y = \frac{1}{4} + C e^{-2x^2}$$

$$y(0) = 1 = \frac{1}{4} + C e^{0} = \frac{1}{4} + C$$

so $C = \frac{3}{4}$. $\quad y = \frac{1}{4} e^{2x^2} + \frac{3}{4}$

5)

a) Find the Taylor series for the function $f(x) = \ln(1+2x)$ (do not quote a known Taylor series).

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n$$

$$f(x) = \ln(1+2x) \qquad f(0) = \ln(1) = 0.$$

$$f'(x) = \frac{1}{1+2x}\cdot 2 \qquad f'(0) = \frac{2}{1} = 2$$

$$f''(x) = \frac{-1}{(1+2x)^{2}}\cdot 2\cdot 2 \qquad f''(0) = -2\cdot 2$$

$$f'''(x) = \frac{(-1)(-2)}{(1+2x)^3}\cdot 2\cdot 2\cdot 2 \qquad f'''(0) = (-1)(-2)(2^3)$$

$$f^{(n)}(x) = \frac{(-2)(-3)(-4)\cdots(-1)(n-1)}{(1+2x)^{n}}\, 2^{n} \qquad f^{(n)}(0) = (-1)^{n-1}\, 2\cdot 3 \dots (n-1)\, 2^n$$

b) Find the radius of convergence for the above series.

$$T(x) = \sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{n-1}(n-1)!\, 2^n x^n$$

Use Ratio Test.

Converges for $\lim_{n\to\infty}\left|\dfrac{(-1)^{n}\frac{2^{n+1}}{n+1}x^{n+1}}{(-1)^{n-1}\frac{2^{n}}{n}x^{n}}\right| < 1$ i.e. $\lim_{n\to\infty} \left|\frac{n+1}{n}\right|\cdot\left|\frac{2}{1}\right||x| < 1$

$|2x| < 1$

$x < \frac{1}{2}$

Radius of conv. is $\frac{1}{2}$.

c) Find the interval of convergence for the series.

$x = \frac{1}{2}$ $\quad T(\frac{1}{2}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ which converges as is alternating harmonic series.

$x = -\frac{1}{2}$ $\quad T(-\frac{1}{2}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\cdot 2^n\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{-1}{n}$ which diverges, as is harmonic series.

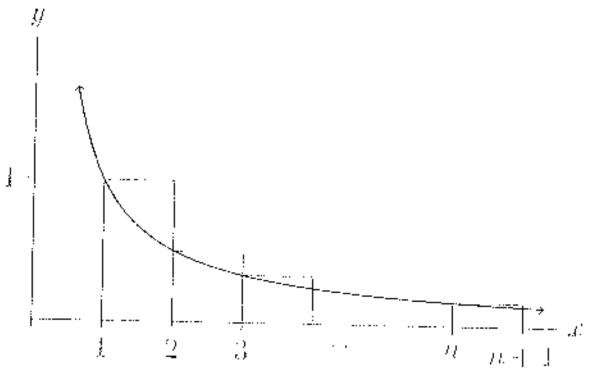
So interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

6) The goal of this problem is to justify the integral test for convergence of a series. Let $\{a_i\}$ be a sequence of positive terms and assume that f is a continuous, non-negative decreasing function with $a_i = f(i)$ for all i.

a) Write L_n for the Riemann sum with left endpoints and $\Delta x = 1$ which approximates the integral

$$\int_1^{n+1} f(x)\,dx,$$

where n is any integer greater than 1. (Here is the picture.)



Express L_n as a finite sum.

$$L_n = \sum_{i=1}^{n} f(i)\cdot 1 = \sum_{i=1}^{n} a_i$$

b) Compare the series $\sum_{i=1}^{n} a_i$ with the integral $\int_1^{n+1} f(x)\,dx$ to find a lower bound for $\sum_{i=1}^{n} a_i$.

From the picture $L_n > \int_1^{n+1} f(x)\,dx$, so

$$\sum_{i=1}^{n} a_i > \int_1^{n+1} f(x)\,dx$$

c) Write R_n for the Riemann sum with right endpoints and $\Delta x = 1$ which approximates the integral

$$\int_1^{n+1} f(x)\,dx,$$

where n is any integer greater than 1. Express R_n as a sum.

$$R_n = \sum_{i=2}^{n+1} f(i) = \sum_{i=2}^{n+1} a_i$$

d) Compare the series with the integral to find an upper bound for $\sum_{i=1}^{n} a_i$.

From the picture, $\sum_{i=2}^{n+1} a_i < \int_1^{n+1} f(x)\,dx$

$$\sum_{i=1}^{n} a_i = a_1 + \sum_{i=2}^{n+1} a_i - a_{n+1} < \int_1^{n+1} f(x)\,dx + a_1 - a_{n+1}$$

e) Use these bounds on $\sum_{i=1}^{n} a_i$ to deduce the statement of the integral test.

As $\sum_{i=1}^{n} a_i > \int_1^{n+1} f(x)\,dx$, so $\lim_{n\to\infty} \sum_{i=1}^{n} a_i \geq \lim_{n\to\infty} \int_1^{n+1} f(x)\,dx$

that is, $\sum_{i=1}^{\infty} a_i \geq \int_1^{\infty} f(x)\,dx$.

Thus, if the improper integral diverges, so does the series.

As $\sum_{i=1}^{n} a_i < \int_1^{n+1} f(x)\,dx + a_1 - a_{n+1}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i \leq \lim_{n\to\infty} \int_1^{n+1} f(x)\,dx + a_1 - \lim_{n\to\infty} (a_{n+1})$$

Thus $\sum_{i=1}^{\infty} a_i \leq \int_1^{\infty} f(x)\,dx + a_1 - 0$ (assuming $\lim_{n\to\infty} a_{n+1} = 0$).

Thus if the improper integral converges, so does the series.