1) (10 points) Find the following integrals.

a) $\int \tan(x)\,dx$.

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$$

let $u = \cos(x)$

$du = -\sin(x)\,dx$

$$= \int -\frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos(x)| + C$$

b) $\int \cos(\sqrt{x})\,dx$

let $u = \sqrt{x}$

$du = \frac{1}{\sqrt{x}} \cdot \frac{1}{2}\,dx$

$2u\,du = dx$

$$= \int \cos(u)\,2u\,du$$

now integrate by parts

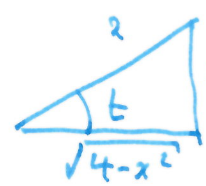
c) $\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$

let $x = 2\sin(t)$

$dx = 2\cos(t)\,dt$

$(4-x^2)^{3/2} = (4\cos^2(t))^{3/2}$

$= 8\cos^3(t)$



$$= \int \frac{4\sin^2(t) \cdot 2\cos(t)\,dt}{8\cos^3(t)}$$

$$= \int \tan^2(t)\,dt$$

$$= \int (\sec^2(t) - 1)\,dt$$

$$= \tan(t) - t + C$$

$$= \frac{x}{\sqrt{4-x^2}} - \arcsin\left(\frac{x}{2}\right) + C$$

2) (10 points) Find all first and second order partial derivatives of the function $f(x, y) = \dfrac{3x}{x^3 - 4y^2}$.

$$f(x,y) = 3x\,(x^3-4y^2)^{-1}$$

$$f_x = 3(x^3-4y^2)^{-1} + 3x(x^3-4y^2)^{-2} \cdot 3x^2$$

$$f_y = 3x(-1)(x^3-4y^2)^{-2}(-8y)$$

$$f_{xx} = -3(x^3-4y^2)^{-2}\,3x^2 + 27x^2(x^3-4y^2)^{-2} + 9x^3(-2)(x^3-4y^2)^{-3}\,3x^2$$

$$f_{xy} = -3(x^3-4y^2)^{-2}(-8y) + 9x^3(-2)(x^3-4y^2)^{-3}(-8y)$$

$$f_{yy} = 24x(x^3-4y^2)^{-2} + \cancel{8}24xy(-2)(x^3-4y^2)^{-3}(-8y)$$

3) (10 points) Find and classify all the critical points of the function $f(x, y) = x\cos(y)$.

$f(x,y) = x\cos(y)$

$f_x = \cos(y) = 0$ when $y = \frac{\pi}{2} + n\pi$

$f_y = -x\sin(y) = 0$ when $x = 0$ or $y = n\pi$.

f_x, f_y both zero when $x = 0$ and $y = \frac{\pi}{2} + n\pi$.

$f_{xx} = 0, \; f_{yy} = -x\cos(y), \; f_{xy} = -\sin(y)$.

$D = f_{xx}f_{yy} - f_{xy}^2 = 0 - \left(-\sin\left(\frac{\pi}{2} + n\pi\right)\right)^2 = -1$

all critical points are saddle points.

4) (10 points) Find the equation of the tangent plane to the surface $z = f(x, y) = e^x \ln(1 + y)$ at the point $(0, 0)$. Use the tangent plane to find an approximation to the value of $f(0.1, -0.1)$.

$f(x,y) = e^x \ln(1+y)$

$f_x = e^x \ln(1+y)$ $\qquad f_x(0,0) = e^0 \ln(1) = 0$

$f_y = e^x \frac{1}{1+y}$ $\qquad f_y(0,0) = e^0 \frac{1}{1+0} = 1.$

$L(x,y) = f_x(x-a) + f_y(y-b) + f(a,b)$

$= 0 + 1 \cdot y + e^0 \ln(1+0)$

$L(x,y) = y + 0.$

$L(0.1, -0.1) =$ ~~0.1 + 1 = 0.9~~ $-0.1.$

5) (10 points) Show that $\lim_{(x,y)\to(0,0)} \frac{x^6}{x^6+3y^2}$ does not exist.

~~Ta~~ Put $x=0$. $\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0}{0+3y^2} = \lim_{y\to 0} 0 = 0.$

Put $y=0$ $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^6}{x^6+0} = \lim_{x\to 0} 1 = 1.$

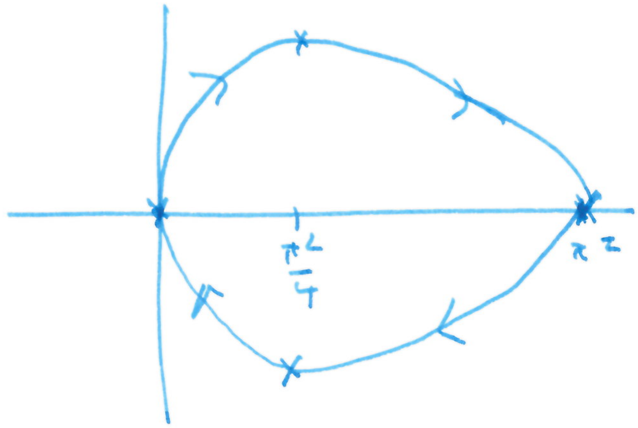
as the two limits are different, $\lim_{(x,y)\to(0,0)} \frac{x^6}{x^6+3x^2}$ does not exist.

6) (10 points) Sketch the curve given by the parametric equations

$$x = t^2, \quad y = \sin(t), \quad -\pi \le t \le \pi$$

Find the area enclosed by this curve to the left of $x = \pi^2$.

t	x	y
$-\pi$	π^2	0
$-\pi/2$	$\pi^2/4$	-1
0	0	0
$\pi/2$	$\pi^2/4$	1
π	π^2	0



area enclosed by curve = 2× area below top half and above x-axis

$$= 2 \times \int_{t=0}^{\pi} y \, dx \qquad \boxed{\begin{array}{l} x = t^2 \\ dx = 2t\,dt \end{array}}$$

$$= 2\int_{t=0}^{\pi} \sin(t)\, 2t \, dt$$

$$= 4\Big[-t\cos(t) + \sin(t)\Big]_0^{\pi} \qquad \text{(use integration by parts)}$$

$$= 4\left(-\pi(-1) + \sin(\pi)\right) - 4(0+0)$$

$$= 4\pi.$$

7) (10 points) Determine whether the following series converge. If the series converges, determine its sum:

(1) $\sum_{n=1}^{\infty} 4^{n-1}$

diverges because $4 > 1$.
(geometric series, or simply $4^{n-1} \not\to 0$)

(2) $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ $= \sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)} = \sum_{n=2}^{\infty} \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$

$$= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}.$$

(For those who find this unsatisfactory, write

$$\sum_{n=2}^{\infty} \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n+1}$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{2}\sum_{k=3}^{\infty} \frac{1}{k}$$

$$= \frac{1}{2}\sum_{k=1}^{2} \frac{1}{k} + \frac{1}{2}\sum_{k=3}^{\infty} \frac{1}{k} - \frac{1}{2}\sum_{k=3}^{\infty} \frac{1}{k}$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\right)$$

$$= 3/2$$

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