MATH 1AA3 Test 1

Name: ____________________
Student No.: ____________________

1)(i)–(v). Multiple choice questions.

Indicate your answers to questions (i)–(v) by circling only ONE of the letters.

(i) [3] Consider the rational function $\frac{x^2-1}{(x^2-4x+4)(x^2+4)}$. Indicate the correct form you would use to express this rational function in partial fractions.

a) $\dfrac{Ax+B}{x^2-4x+4}+\dfrac{Cx+D}{x^2+4}$

b) $\dfrac{A}{x-2}+\dfrac{B}{x-2}+\dfrac{C}{x-2}+\dfrac{D}{x+2}$

c) $\dfrac{A}{x-2}+\dfrac{Bx+C}{(x-2)^2}+\dfrac{D}{x^2+4}$

d) $\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}+\dfrac{Cx+D}{x^2+4}$

e) $\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}+\dfrac{C}{x^2+4}$

f) $\dfrac{A(x-1)}{x-2}+\dfrac{B(x+1)}{x-2}+\dfrac{Cx+D}{x^2+4}$

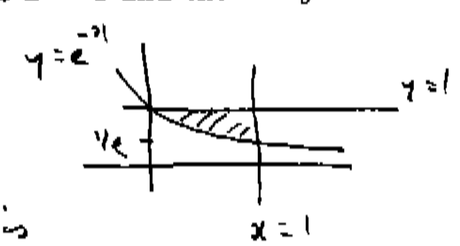
$x^2-4x+4=(x-2)^2$

x^2+4 is irreducible.

$$\frac{x^2-1}{(x^2-4x+4)(x^2+4)}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{Cx+D}{x^2+4}.$$

d.

(ii) [3] Indicate the integral that you would use to find the volume of the solid formed by rotating the region bounded by the graph of $y=e^{-x}$, the line $x=1$ and the line $y=1$ around the line $x=1$.



a) $\int_0^1 \pi(1-e^{-y})^2\,dy$

b) $\int_{1/e}^1 \pi(1-e^{-y})^2\,dy$

c) $\int_0^1 \pi(1-\ln(y))^2\,dy$

d) $\int_{1/e}^1 \pi(1-\ln(y))^2\,dy$

e) $\int_0^1 \pi(1-\ln(\frac{1}{y}))^2\,dy$

f) $\int_{1/e}^1 \pi(1-\ln(\frac{1}{y}))^2\,dy$

cross-sectional area is

$\pi(1-x)^2$

$y=e^{-x} \Rightarrow x=-\ln(y)=\ln\left(\frac{1}{y}\right)$

$$\text{volume}=\int_{1/e}^{1}\pi\left(1-\ln\left(\tfrac{1}{y}\right)\right)^2 dy$$

e

Continued on next page

MATH 1AA3 Test 1

Name:__________________
Student No.: ______________

(iii) [3] Indicate the integral which is equal to $\int \frac{\sqrt{x^2+1}}{x}\,dx$ after applying a trigonometric substitution.

a) $\int \frac{\sec(\theta)}{\tan(\theta)}\,d\theta$

b) $\int \frac{\sec^2(\theta)}{\tan^2(\theta)}\,d\theta$

c) $\int \frac{\sec^3(\theta)}{\tan(\theta)}\,d\theta$

d) $\int \frac{\sec^3(\theta)}{\tan^2(\theta)}\,d\theta$

e) $\int \frac{\cos^2(\theta)}{\sin(\theta)}\,d\theta$

f) $\int \frac{\cos(\theta)}{\sin(\theta)}\,d\theta$

Let $x = \tan(\theta)$

$x^2+1 = \tan^2(\theta)+1 = \sec^2\theta$

$\frac{dx}{d\theta} = \sec^2\theta$

$\int \frac{\sqrt{x^2+1}}{x}\,dx = \int \frac{\sqrt{\sec^2\theta}}{\tan\theta}\sec^2\theta\,d\theta$

$= \int \frac{\sec^3\theta}{\tan\theta}\,d\theta$ c

(iv) [3] Indicate the integral which is equal to $\int x^2\ln(x)\,dx$.

a) $\frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^2\,dx$

b) $\frac{1}{3}x\ln(x) - \int \frac{1}{3}x^2\,dx$

c) $x^2\ln(x) - \int x^2\,dx$

d) $\frac{1}{3}x^2 - \int \frac{1}{x^3}\ln(x)\,dx$

e) $\frac{1}{3}x^2 - \int \frac{1}{3}x\ln(x)\,dx$

f) $x^2 - \int x^2\ln(x)\,dx$

Integration by parts:

$u = \ln x$ $\quad dv = x^2\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = \frac{1}{3}x^3$

$\int x^2\ln(x)\,dx = \frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^3\cdot\frac{1}{x}\,dx$

a

(v) [3] Indicate the limit or limits which can be correctly used to calculate $\int_0^\infty \ln(x)\,dx$.

a) $\lim_{t\to\infty}\int_0^t \ln(x)\,dx$

b) $\lim_{t\to 1^-}\int_0^t \ln(x)\,dx + \lim_{t\to 1^+}\int_1^\infty \ln(x)\,dx$

c) $\lim_{t\to 0^+}\int_t^1 \ln(x)\,dx + \lim_{t\to\infty}\int_1^t \ln(x)\,dx$

d) $\lim_{t\to 0^+}\int_t^\infty \ln(x)\,dx$

$\ln(x)$ is undefined at 0.

c

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MATH 1AA3 Test 1

Name:

Student No.:

In the following questions, you must show all your work to receive full credit. You may use integrals from the Table of Formulas, but you should refer to them by number.

2) Calculate the following integrals or explain why the integral does not exist.

a)[5] $\int \sin^2(x)\cos^2(x)\, dx$

Substitute $\sin^2(x) = \frac{1}{2}(1+\cos(2x))$

and $\cos^2(x) = \frac{1}{2}(1-\cos(2x))$

$= \int \frac{1}{4}(1-\cos^2(2x))\, dx$

Substitute $\cos^2(2x) = \frac{1}{2}(1-\cos(4x))$

$= \int \left(\frac{1}{8} - \frac{1}{8}\cos(4x)\right) dx$

$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$

b)[6] $\int \sin(\sqrt{x})\, dx$

Substitute $t = \sqrt{x}$

$t^2 = x$

$2t\,dt = dx$

$= \int 2t\sin(t)\, dt$

then integrate by parts

$u = 2t$ $\quad dv = \sin(t)\,dt$

$du = 2dt$ $\quad v = -\cos(t)$

$= -2t\cos(t) - \int -2\cos(t)\, dt$

$= -2t\cos(t) + 2\sin(t) + C$

$= -2\sqrt{x}\cos(\sqrt{x}) + 2\sin(\sqrt{x}) + C.$

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MATH 1AA3 Test 1

Name:________________________

Student No.: ____________________

3) Calculate the following integrals or explain why the integral does not exist.

a)[6] $\int_0^2 \frac{1}{x-1}\,dx$

$$= \lim_{t\to 1^-}\int_0^t \frac{1}{x-1}\,dx + \lim_{t\to 1^+}\int_t^2 \frac{1}{x-1}\,dx$$

$$\lim_{t\to 1^-}\int_0^t \frac{1}{x-1}\,dx = \lim_{t\to 1^-}\Big[\ln|x-1|\Big]_0^t$$

$$= \lim_{t\to 1^-}\left(\ln|t-1| - \ln(1)\right)$$

$$= -\infty.$$

One the limits does not exist, hence the integral does not exist.

b)[6] $\int \frac{3x-7}{x^2-4x+3}\,dx$

$$\frac{3x-7}{x^2-4x+3} = \frac{A}{x-3} + \frac{B}{x-1}$$

$$3x-7 = A(x-1) + B(x-3)$$

$x=1$: $\quad -4 = -2B \qquad \underline{B=2}$

$x=3$: $\quad 2 = 2A \qquad \underline{A=1}$

$$\int \frac{3x-7}{x^2-4x+3}\,dx = \int \left(\frac{1}{x-3} + \frac{2}{x-1}\right) dx$$

$$= \ln|x-3| + 2\ln|x-1| + C$$

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MATH 1AA3 Test 1 Name:______________
Student No.: ______________

4) [6] Calculate the average value of the function $f(x) = e^x \sin(x)$ on the interval $[0, \pi]$.

avg value $= \frac{1}{\pi - 0} \int_0^\pi e^x \sin(x)\,dx$

$\int e^x \sin(x)\,dx = -e^x \cos(x) + \int \cos(x) e^x\,dx$ IP twice

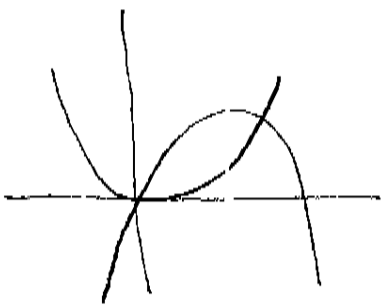
$= -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x)\,dx$

Hence $\int e^x \sin(x)\,dx = \frac{1}{2} e^x (\sin(x) - \cos(x)) + C$.

avg. value $= \frac{1}{\pi} \left[\frac{1}{2} e^x (\sin(x) - \cos(x)) \right]_0^\pi$

$= \frac{1}{2\pi} (e^\pi + 1)$.

5) [6] Calculate the volume of the solid formed by rotating the area enclosed by the graphs of $y = x^2$ and $y = 2x - x^2$ around the x-axis.



Graphs meet when $x^2 = 2x - x^2$

$0 = 2x - 2x^2$

$x = 0$ or $x = 1$

cross-sectional area is annulus,

area $= \pi (R^2 - r^2)$

$= \pi ((2x - x^2)^2 - (x^2)^2)$

volume $= \pi \int_0^1 ((2x - x^2)^2 - (x^2)^2)\,dx$

$= \pi \int_0^1 (4x^2 - 4x^3)\,dx = \pi/3$.

THE END