Section 6.1 Question 6

Find a formula for $0 \cdot 1 \cdot 2 + 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2)$, for $n \in \mathbb{N}$, and prove that your formula is correct. (Hint: Compare this question to exercises 1 and 5, and try to guess the formula.)

Note: Exercise 1 asks you to prove that $0 + 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

Exercise 5 asks you to prove that $0 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

Answer: From exercises 1 and 5 we can guess that the formula we want will be $\frac{n(n+1)(n+2)(n+3)}{4}$. Let us now attempt to prove this by induction.

Base Case: $n = 0$. It is obvious that $0 \cdot 1 \cdot 2 = 0 = \frac{0 \cdot 1 \cdot 2 \cdot 3}{4}$, so the base case is finished.

Induction Step: Assume that $0 \cdot 1 \cdot 2 + 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$. We will prove that $0 \cdot 1 \cdot 2 + 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2) + (n+1)(n+2)(n+3) = \frac{(n+1)(n+2)(n+3)(n+4)}{4}$.

Using our inductive assumption, we know that

$$
\begin{aligned}
0 \cdot 1 \cdot 2 + 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2) + (n+1)(n+2)(n+3) \\
= \tfrac{n(n+1)(n+2)(n+3)}{4} + (n+1)(n+2)(n+3) \\
= \tfrac{n(n+1)(n+2)(n+3)}{4} + \tfrac{4(n+1)(n+2)(n+3)}{4} \\
= \tfrac{(n+4)(n+1)(n+2)(n+3)}{4}
\end{aligned}
$$

as desired. Therefore, by induction, the formula is proven to be the one we wanted.

Section 6.1 Question 12

Prove that for all integers a and b and all $n \in \mathbb{N}$, $(a-b)|(a^n - b^n)$. (Hint: Let a and b be arbitrary integers and then prove by induction that $\forall n \in \mathbb{N}[(a-b)|(a^n - b^n)]$. For the induction step, you must relate $a^{n+1} - b^{n+1}$ to $a^n - b^n$. You might find it useful to start by completing the following equation: $a^{n+1} - b^{n+1} = a(a^n - b^n) + \underline{?}$.)

Answer: Following the hint, we will begin by completing the equation given. $a^{n+1} - b^{n+1} = a(a^n - b^n) + a \cdot b^n - b^{n+1} = a(a^n - b^n) + (a-b)b^n$. Now, Given any integers a and b, we perform an induction on n:

Base Case: $n = 1$. It is obvious that $(a-b)|(a-b)$, so the base case is finished.

Induction Step: Assume that $(a-b)|(a^n - b^n)$. We will prove that $(a-b)|(a^{n+1} - b^{n+1})$. Since $(a-b)|(a^n - b^n)$, there is some $k \in \mathbb{Z}$ such that $k(a-b) = a^n - b^n$. From the equation we completed above, we know that

$$
\begin{aligned}
a^{n+1} - b^{n+1} &= a(a^n - b^n) + (a-b)b^n \\
&= ak(a-b) + (a-b)b^n \\
&= (a-b)(ak + b^n)
\end{aligned}
$$

So $(a-b)|(a^{n+1} - b^{n+1})$ as desired. By induction, therefore, the claim is proven for all n. Since a and b were chosen arbitrarily, we can conclude that this is true for all integers a and b.

Section 6.4 Question 4

The Martian monetary system uses colored beads instead of coins. A blue bead is worth 3 Martian credits, and a red bead is worth 7 Martian credits. Thus, three blue beads are worth 9 credits, and a blue and red bead together are worth 10 credits, but no combination of blue and red beads is worth 11 credits. Prove that for all $n \geq 12$, there is some combination of blue and red beads that is worth n credits.

Answer: Let a represent the number of blue beads and b the number of red beads in a combination, and let n represent the number of credits. We obtain the following equation: $n = 3a + 7b$. We want to prove that for all $n \geq 12$, there are $a, b \in \mathbb{N}$ such that $n = 3a + 7b$, and we will prove this by induction.

Base Cases: $n = 12$. Clearly $12 = 3 \cdot 4 + 7 \cdot 0$.

$n = 13$. Clearly $13 = 3 \cdot 2 + 7 \cdot 1$.

$n = 14$. Clearly $14 = 3 \cdot 0 + 7 \cdot 2$.

Induction Step: Assume that for all $12 \leq k < n$ there are $x, y \in \mathbb{N}$ such that $k = 3x + 7y$. We will now prove that there are $a, b \in \mathbb{N}$ such that $n = 3a + 7b$. By the induction hypothesis we know that there are $x, y \in \mathbb{N}$ such that $n - 3 = 3x + 7y$. Therefore $n = n - 3 + 3 = 3x + 7y + 3 = 3(x + 1) + 7y$, so choosing $a = x + 1$ and $b = y$ we see that $n = 3a + 7b$. Therefore by induction, the claim is true for all $n \geq 12$.

Alternate Answer: Here is a more direct proof of this claim which does not use induction. Given $n \geq 12$, the division algorithm tells us that there are unique $q, r \in \mathbb{N}$ such that $n = 3q + r$ and $0 \leq r < 3$. We now consider three cases:

Case 1: If $r = 0$ then $n = 3q$ and so $n = 3q + 7 \cdot 0$ as required.

Case 2: If $r = 1$ then $n = 3q + 1 = 3(q - 2) + 7$ so $n = 3(q - 2) + 7 \cdot 1$.

Case 3: If $r = 2$ then $n = 3q + 2 = 3(q - 4) + 14$ so $n = 3(q - 4) + 7 \cdot 2$.

We are not quite finished, since we need to prove that $q - 2$ and $q - 4$ in cases 2 and 3 respectively are still natural numbers; we will prove this by contradiction. Suppose that $q < 4$, then $n < 3 \cdot 4 = 12$, contradicting our initial information that $n \geq 12$, therefore $q \geq 4$ and so $q - 2$ and $q - 4$ are nonnegative.