Section 2.1 Question 8

Are these statements true or false? The universe of discourse is $\mathbb{R}$.

(a) $\forall x \exists y(2x - y = 0)$
Answer: True.
Given $x \in \mathbb{R}$, pick $y = 2x$, then $2x - y = 0$.

(b) $\exists y \forall x(2x - y = 0)$
Answer: False.
Such a y would have to be equal to $0 = 2 \cdot 0$ and $4 = 2 \cdot 2$ at the same time, clearly impossible.

(c) $\forall x \exists y(x - 2y = 0)$
Answer: True.
Given $x \in \mathbb{R}$, pick $y = \frac{x}{2}$, then $x - 2y = 0$.

(d) $\forall x(x < 10 \rightarrow \forall y(y < x \rightarrow y < 9))$
Answer: False.
Pick $x = 9.5$, then $x < 10$ is true. Pick $y = 9.25$, then $y < x$ is true but $y < 9$ is false.

(e) $\exists y \exists z(y + z = 100)$
Answer: True.
Pick $y = z = 50$, then $y + z = 100$.

(f) $\forall x \exists y(y > x \wedge \exists z(y + z = 100))$
Answer: True.
Given $x \in \mathbb{R}$, pick $y = x + 1$ and $z = 99 - x$. Then $y > x$ and
$y + z = x + 1 + 99 - x = 100$.

Section 2.2 Question 10

(a) Show that $\exists x \in A\ P(x) \vee \exists x \in B\ P(x)$ is equivalent to $\exists x \in (A \cup B)\ P(x)$.
Solution: $\exists x \in (A \cup B)\ P(x)$
is equivalent to $\exists x(x \in (A \cup B) \wedge P(x))$
is equivalent to $\exists x((x \in A \vee x \in B) \wedge P(x))$ (by the definition of $\cup$)
is equivalent to $\exists x((x \in A \wedge P(x)) \vee (x \in B \wedge P(x)))$ (distributive law)
is equivalent to $\exists x \in A\ P(x) \vee \exists x \in B\ P(x)$

(b) Is $\exists x \in A\ P(x) \wedge \exists x \in B\ P(x)$ equivalent to $\exists x \in (A \cap B)\ P(x)$? Explain.
These two statements are not equivalent and there and many counterexamples which can demonstrate this. One such counterexample:
Let A be the set of prime numbers and B be the set of even numbers, and let $P(x)$ stand for the property "$x > 10$". Clearly $\exists x \in A\ P(x)$ is true (pick $x = 11$) and $\exists x \in B\ P(x)$ is true (pick $x = 12$), so $\exists x \in A\ P(x) \wedge \exists x \in B\ P(x)$ is also true. However $A \cap B = \{2\}$ and so $\exists x \in (A \cap B)\ P(x)$ is false.

Section 2.3 Question 6

Let $I = \{2, 3, 4, 5\}$ and for each $i \in I$ let $A_i = \{i, i + 1, i - 1, 2i\}$.

(a) List the elements of all the sets A_i for $i \in I$.

Answer:
$A_2 = \{2, 3, 1, 4\}$
$A_3 = \{3, 2, 4, 6\}$
$A_4 = \{4, 3, 5, 8\}$
$A_5 = \{5, 4, 6, 10\}$

(b) Find $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$.

Answer:

$\bigcap_{i \in I} A_i = \{2, 3, 1, 4\} \cap \{3, 2, 4, 6\} \cap \{4, 3, 5, 8\} \cap \{5, 4, 6, 10\}$
$= \{2, 3, 4\} \cap \{4, 5\}$
$= \{4\}$
$\bigcup_{i \in I} A_i = \{2, 3, 1, 4\} \cup \{3, 2, 4, 6\} \cup \{4, 3, 5, 8\} \cup \{5, 4, 6, 10\}$
$= \{1, 2, 3, 4, 6\} \cup \{3, 4, 5, 6, 8, 10\}$
$= \{1, 2, 3, 4, 5, 6, 8, 10\}$