## Section 3.5 Question 12

(a) Prove that for all real numbers $a$ and $b$, $|a| \leq b$ iff $-b \leq a \leq b$.

**Answer**: ($\rightarrow$): Suppose $|a| \leq b$. We aim to prove that $-b \leq a$ and $a \leq b$.

Clearly $a \leq |a|$ for all $a$, so $a \leq b$ is true.

*Case 1*: If $a \geq 0$, $a \geq -b$ is trivial since $b \geq 0$ as well.

*Case 2*: If $a < 0$, then $|a| = -a$ and so $|a| \leq b$ tells us $-a \leq b$ and so $-b \leq a$ (by multiplying both sides of the inequality by $-1$).

($\leftarrow$): Suppose $-b \leq a \leq b$. $-b \leq a$ tells us that $-a \leq b$ (multiply both sides by $-1$). Since $a \leq b$ and $-a \leq b$, we get that $|a| \leq b$.

(b) Prove that for any real number $x$, $-|x| \leq x \leq |x|$.

**Answer**: Let $a = x$ and $b = |x|$. Clearly $|x| = a \leq b = |x|$, so part (a) above tells us that $-b \leq a \leq b$, that is $-|x| \leq x \leq |x|$.

(c) Prove that for all real numbers $x$ and $y$, $|x + y| \leq |x| + |y|$.

**Answer**: Part (b) above tells us that $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Adding these together we get that $-|x| - |y| \leq x + y \leq |x| + |y|$. Using part (a), this tells us that $|x + y| \leq |x| + |y|$ as required.

## Section 4.2 Question 9

Suppose $R$ and $S$ are relations from $A$ to $B$. Must the following statements be true? Justify your answers with proofs or counterexamples.

(a) $R \subseteq \text{Dom}(R) \times \text{Ran}(R)$.

**Answer**: True. Given any $(a, b) \in R$, we know that $a \in \text{Dom}(R) = \{x : \exists y((x, y) \in R)\}$ and $b \in \text{Ran}(R) = \{y : \exists x((x, y) \in R)\}$, therefore $(a, b) \in \text{Dom}(R) \times \text{Ran}(R)$. This tells us, by definition, that $R \subseteq \text{Dom}(R) \times \text{Ran}(R)$.

(b) If $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$.

**Answer**: True. Suppose that $R \subseteq S$ and that $(a, b) \in R^{-1}$. Since $(a, b) \in R^{-1}$ we know that $(b, a) \in R$, so our initial assumption tells us that $(b, a) \in S$. By the definition of $S^{-1}$, this tells us that $(a, b) \in S^{-1}$ as required.

(c) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

**Answer**: True. We will first prove that $(R \cup S)^{-1} \subseteq R^{-1} \cup S^{-1}$.

Given any $(a, b) \in (R \cup S)^{-1}$, we know that $(b, a) \in R \cup S$ so either $(b, a) \in R$ or $(b, a) \in S$.

*Case 1*: If $(b, a) \in R$ then $(a, b) \in R^{-1}$ and so $(a, b) \in R^{-1} \cup S^{-1}$ as required.

*Case 2*: If $(b, a) \in S$ then $(a, b) \in S^{-1}$ and so $(a, b) \in R^{-1} \cup S^{-1}$ as required.

**Note**: Case 2 above is identical to Case 1, except that all '$S$'s are replaced by '$R$'s and vice versa. In this sort of scenario we would usually omit Case 2 and simply say that Case 1 is sufficient to prove our claim by the symmetry of the argument.

Next we must prove that $R^{-1} \cup S^{-1} \subseteq (R \cup S)^{-1}$.

Clearly $R \subseteq R \cup S$ and $S \subseteq R \cup S$. Using part (b) above, this tells us that $R^{-1} \subseteq (R \cup S)^{-1}$ and $S^{-1} \subseteq (R \cup S)^{-1}$, therefore $R^{-1} \cup S^{-1} \subseteq (R \cup S)^{-1}$ as required.

Since $(R \cup S)^{-1} \subseteq R^{-1} \cup S^{-1}$ and $R^{-1} \cup S^{-1} \subseteq (R \cup S)^{-1}$, we conclude that $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.