Section 4.3 Question 19

Consider the Following putative theorem:

Theorem? Suppose R is a relation on A and define a relation S on $\mathcal{P}(A)$ as follows:

$S = \{(X, Y) \in \mathcal{P}(A) \times \mathcal{P}(A) | \exists x \in X \exists y \in Y (xRy)\}$.

If R is transitive, then so is S.

(a) What's wrong with the following proof of the theorem?

Proof: Suppose R is transitive. Suppose $(X, Y) \in S$ and $(Y, Z) \in S$. Then by the definition of S, $\underline{xRy \text{ and } yRz}$, where $x \in X$, $y \in Y$, and $z \in Z$. Since xRy, yRz and R is transitive, xRz, but then since $x \in X$ and $z \in Z$, it follows from the definition of S that $(X, Z) \in S$. Thus, S is transitive.

Answer: The underlined portion of the proof is incorrect because we cannot assume that the same $y \in Y$ witnesses both $(X, Y) \in S$ and $(Y, Z) \in S$.

(b) Is the theorem correct? Justify your answer with either a proof or a counterexample.

Answer: The theorem is incorrect and this can be seen in the following somewhat unnatural counterexample.

Let $A = \mathbb{N}$ and define $R = \{(x, y) \in \mathbb{N}^2 | x < y$ and either x and y are both even or both odd$\}$. R is a transitive relation on $\mathbb{N}$.

Then clearly $(1, 3) \in R$, $(2, 4) \in R$, and $(1, 4) \notin R$, so $(\{1\}, \{2, 3\}) \in S$ and $(\{2, 3\}, \{4\}) \in S$, but $(\{1\}, \{4\}) \notin S$ so S is not transitive.

A more natural counterexample was provided by a student, paraphrased here:

Let $A = \mathbb{N}$, $R = \{(x, y) \in \mathbb{N}^2 | \exists k \in \mathbb{N}(kx = y)\}$ (i.e. R is the relation "x divides y") and pick $X = \{4\}$, $Y = \{2, 8\}$, and $Z = \{6\}$ then $(X, Y) \in S$ since $4|8$ and $(Y, Z) \in S$ since $2|6$ but $(X, Z) \notin S$ since $4 \not| 6$.

Section 4.6 Question 14

Suppose that $B \subseteq A$, and define a relation R on $\mathcal{P}(A)$ as follows:

$R = \{(X, Y) \in \mathcal{P}(A) \times \mathcal{P}(A) | (X \Delta Y) \subseteq B\}$.

(a) Prove that R is an equivalence relation on $\mathcal{P}(A)$.

Answer: In order to establish that R is an equivalence relation, we must prove that it is reflexive, symmetric and transitive. Also, recall that

$X \Delta Y = (X \cup Y) \setminus (X \cap Y)$.

Reflexivity: Given any $X \in \mathcal{P}(A)$, $X \cup X = X \cap X$ so $X \Delta X = \emptyset$. Therefore $X \Delta X = \emptyset \subseteq B$ so $(X, X) \in R$.

Symmetry: Given any $(X, Y) \in R$, we know that $X \Delta Y \subseteq B$. Since Δ is symmetric, however, this tells us that $Y \Delta X \subseteq B$ and so $(Y, X) \in R$ as well.

Transitivity: Given $(X, Y), (Y, Z) \in R$, we know that $X \Delta Y \subseteq B$ and $Y \Delta Z \subseteq B$. We need to prove that $X \Delta Z \subseteq B$. Pick any $a \in X \Delta Z$, we need to prove that $a \in B$. Since $a \in X \Delta Z$, either $a \in X$ or $a \in Z$ but not both.

Case 1: If $a \notin Y$ then this tells us that $a \in X \rightarrow a \in X \Delta Y \subseteq B$ and $a \in Z \rightarrow a \in Y \Delta Z \subseteq B$, either way $a \in B$.

Case 2: If $a \in Y$ then $a \in X \rightarrow a \notin Z \rightarrow a \in Y \Delta Z \subseteq B$ and $a \in Z \rightarrow a \notin X \rightarrow a \in X \Delta Y \subseteq B$. Again we get that $a \in B$ either way.

Since $a \in B$ in all cases, we get that $X \Delta Z \subseteq B$ and so $(X, Z) \in R$ proving the transitivity of R.

(b) Prove that for every $X \in \mathcal{P}(A)$ there is exactly one $Y \in [X]_R$ such that $Y \cap B = \emptyset$.

Answer: given $X \in \mathcal{P}(A)$, there are two things for us to prove here; we need to prove that such a Y exists and we must prove that it is unique.

Existence: Let $Y = X \setminus B$. Then $X \Delta Y = X \cap B \subseteq B$ so $Y \in [X]_R$ and $Y \cap B = (X \setminus B) \cap B = \emptyset$.

Uniqueness: Suppose that $Y, Z \in [X]_R$ and $Y \cap B = Z \cap B = \emptyset$, we want to prove that $Y = Z$. Since $Y, Z \in [X]_R$ we know that $(Y, X), (Z, X) \in R$ so by symmetry and transitivity $(Y, Z) \in R$. Therefore $Y \Delta Z \subseteq B$. However, $Y \Delta Z \subseteq Y \cup Z$ and $(Y \cup Z) \cap B = \emptyset$ so $Y \Delta Z \subseteq \emptyset$, hence $Y \cup Z \subseteq Y \cap Z$, therefore $Y \cup Z = Y \cap Z = Y = Z$, which is what we needed to show.