Section 5.2 Question 9

Suppose $f : A \to B$ and $g : B \to C$.

(a) Prove that if f is onto and g is not one-to-one, then $g \circ f$ is not one-to-one.

Answer: Suppose that f is onto and g is not one-to-one. We can find elements in B to witness the fact that g is not one-to-one, namely we can find $x, y \in B$ with $x \neq y$ and $g(x) = g(y)$. Since f is onto, we can then find $a, b \in A$ with $f(a) = x$ and $f(b) = y$. Since $x \neq y$, the definition of a function tells us that $a \neq b$. Therefore, we have that $(g \circ f)(a) = g(f(a)) = g(x) = g(y) = g(f(b)) = (g \circ f)(b)$, but $a \neq b$ and so $g \circ f$ is not one-to-one.

(b) Prove that if f is not onto and g is one-to-one, then $g \circ f$ is not onto.

Answer: Suppose that f is not onto and g is one-to-one. We can find an $x \in B$ which witnesses the fact that f is not onto, i.e. $x \notin Ran(f)$. In order to prove that $g \circ f$ is not onto, we now want to show that for any $a \in A$, $(g \circ f)(a) \neq g(x)$, demonstrating that $Ran(g \circ f) \subsetneq C$. Given any $a \in A$, we know that $f(a) \neq x$ since x is not in the range of f. Since g is one-to-one, $f(a) \neq x$ tells us that $g(f(a)) \neq g(x)$ as well, which is what we were trying to prove.

Section 5.2 Question 15

Suppose R is an equivalence relation on A and let $g : A \to A/R$ be defined by the formula $g(x) = [x]_R$, as in exercise 17 in Section 5.1.

(a) Show that g is onto.

Answer: Pick an arbitrary $x \in A/R$. Since x is an equivalence class of R, it is a nonempty subset of A and so we can pick $y \in x$. Therefore $g(y) = [y]_R = x$ and so $x \in Ran(g)$.

(b) Show that g is one-to-one iff $R = i_A$.

Answer: ($\rightarrow$): Suppose that g is one-to-one. We want to prove that $R = i_A$, that is that xRy iff $x = y$. Given any $x, y \in A$ with xRy, we know that $g(x) = [x]_R = [y]_R = g(y)$. Since g is one-to-one this then tells us that $x = y$, which is what we needed (remembering that the other direction of the iff we were trying to prove is part of the definition of an equivalence relation).

($\leftarrow$): Suppose that $R = i_A$. We want to prove that g is one-to-one, that is we want to prove that given any $x, y \in A$, $g(x) = g(y)$ implies that $x = y$. So, given any $x, y \in A$ with $g(x) = g(y)$ we know that $[x]_R = g(x) = g(y) = [y]_R$ and so xRy is true. Since $R = i_A$, this tells us that $x = y$ is also true, thus proving the claim.

Section 5.3 Question 6

Let $A = \mathbb{R} \setminus \{2\}$, and let f be the function with domain A defined by the formula $f(x) = \frac{3x}{x-2}$.

(a) Show that f is a one-to-one, onto function from A to B for some set $B \subseteq \mathbb{R}$. What is the set B?

Answer: Since $Ran(f) \subseteq \mathbb{R}$, we can choose $B = Ran(f)$ and it is obvious that f is onto B. In order to prove that f is one-to-one, pick arbitrary $x, y \in A$ with $f(x) = f(y)$ and we will show that $x = y$. By our assumption, $f(x) = \frac{3x}{x-2} = \frac{3y}{y-2} = f(y)$ and so $3x(y-2) = 3y(x-2)$. Expanding and collecting everything to one side of the equation we get that $3xy - 6x - 3xy + 6y = 0$, i.e. $6x = 6y$ so $x = y$, proving our claim.

In order to determine what the set B is, we choose arbitrary $y \in \mathbb{R}$ and find conditions under which there is an $x \in A$ with $f(x) = y$. Specifically, we need to isolate x in the equation $f(x) = y$ and eliminate the cases where the result is undefined in y.

$$\begin{aligned} \tfrac{3x}{x-2} &= y \\ 3x &= y(x-2) \\ 3x &= xy - 2y \\ (y-3)x &= 2y \\ x &= \tfrac{2y}{y-3} \end{aligned}$$

This equation is undefined when $y = 3$ but defined everywhere else, hence $B = \mathbb{R} \setminus \{3\}$.

(b) Find a formula for $f^{-1}(x)$.

Answer: In the previous part, we calculated that $f(x) = y$ iff $x = \frac{2y}{y-3}$, therefore $f^{-1}(y) = \frac{2y}{y-3}$ and so $f^{-1}(x) = \frac{2x}{x-3}$.