Math 1C03 Problem Sheet 1 Solutions

1) (i) The number p is prime if it is not divisible by any integer except itself and 1.

(ii) Two numbers a and b are relatively prime if there is no integer except 1 which divides both of them.

(iii) ~~If the 2 pair of primes~~
Let p and q be prime numbers. If we allow $p = q$, then they are not relatively prime, as both are divisible by p.
If we do not allow $p = q$ then they are relatively prime. For we know that the only number apart from 1 that divides p is p. Since $q \neq p$, we know that $p \nmid q$. So there is no number that divides both of them.
The correct statement of the proposition is:
~~Any pair of~~ Any two distinct prime numbers are relatively prime.

2) (i) $a|b$ if a,b are positive integers and there is a positive integer n such that $b = an$.

(ii) Suppose $a^2 = 2b^2$, so $a \cdot a = 2 \cdot b \cdot b$.
By Theorem 2.5.3 (stated in class, also in text) as 2 is prime and ~~$2 | 2bb$ and~~ $2 | a.a$ either $2|a$ or $2|a$. So $2|a$, that is, there is n such that $a = 2n$. The equation ~~becomes~~ $a^2 = 2b^2$ becomes

$$(2n)(2n) = 2bb .$$

Dividing by 2 gives $n \cdot 2 \cdot n = b \cdot b$.
Again, $2 | b \cdot b$, so $2|b$.
We have shown $2|a$ and $2|b$, as required.

(iii) Assume $\sqrt{2}$ is a rational number.
So $\sqrt{2} = \frac{m}{n}$ where m, n are positive integers, and we may assume m and n have no common factors.

Squaring: $2 = \frac{m^2}{n^2}$ or $2n^2 = m^2$.
By (ii) $2|n$ and $2|m$. This is a contradiction, as we assume m and n have no common factors.

(iv) Because the assumption that $\sqrt{2}$ is rational produced a contradiction, the assumption must be false. That is, $\sqrt{2}$ is irrational.

(v) Use the same method to prove $\sqrt{3}$ is irrational.

Suppose $\sqrt{3}$ is rational, so $\sqrt{3} = \frac{m}{n}$, where m, n have no common factors.

Then $3 = \frac{m^2}{n^2}$ or $3n^2 = m^2$.

Then $3 \mid m^2$, and as 3 is prime, $3 \mid m$.

So $m = 3m_1$ and $3n^2 = (3m_1)^2$.

Hence $n^2 = 3m_1^2$

That is, $3 \mid n^2$, hence $3 \mid n$.

But we assumed that m, n have no common factors, so we have a contradiction.

Thus $\sqrt{3}$ cannot be rational.

Note: Where does this argument break down if we try to prove that $\sqrt{4}$ is not rational?

3) i), ii), iii) look at the list of primes.

iv) A (variable) quantity is <u>arbitrarily large</u> if, for any positive integer N, there is an instance of the quantity which is bigger than N.

(v) Another way to say that the gaps between consecutive prime numbers is arbitrarily large is to say that there are arbitrarily long sequences of composite numbers.

The argument given in class showed that the sequence of numbers

$$\{(N+1)!+2,\ (N+1)!+3,\ (N+1)!+4, \dots, (N+1)!+N+1\}$$

is a sequence of length N of composite numbers.

4) $E = \{2n : n \in \mathbb{N}, n \neq 0\}$

$a \mid_E b$ if there is $q \in E$ st. $b = aq$.

(we can say: $a \mid_E b$ if $\frac{b}{a}$ is an even number)

i) $2 \mid_E 12$ because $2, 12 \in E$, $12 = 2 \cdot 6$ and $6 \in E$.

ii) $2 \nmid_E 10$ because $10 = 2 \cdot 5$ and $5 \notin E$.

iii) p is prime in E if there is no number in E that divides ~~..~~ p in E.

iv) Claim Let $e = 2n$ be any element of E. e is prime in E if and only if n is odd.

To prove the claim, there are two assertions to verify.

1) If $e = 2n$ is prime in E then n is odd.

2) If n is odd then $e = 2n$ is prime in E.

To prove 1), assume $e = 2n$ is prime in E. We want to show that n is odd, so suppose not and derive a contradiction. If n is not odd then n is even, so $n = 2n'$. Then $e = 2n = 2 \cdot 2n'$.

But then $\frac{e}{2} = 2n'$ which is in E, so $2 \mid_E e$. This contradicts the assumption that e is prime in E.

To prove 2), assume n is odd. To prove that $e = 2n$ is prime in E, again work by contradiction. So suppose that e is not prime in E. Then there is some even number that divides e, say $e = 2a \cdot 2b$. Then $2n = 2a \cdot 2b$, or $n = a \cdot 2b$. Thus n is even. This is a contradiction.

v) Idea: factor out as many 2's as possible. Let $e \in E$. We can write $e = 2^k n$, where n is odd, for some $k \geq 1$. Now write

$e = 2^{k-1}(2n)$.

2 itself is prime in E, so 2^{k-1} is a product of primes

$2n$ is prime in E by iv) as n is odd.

Thus we have written e as a product of numbers that are prime in E.

vi) $60 = (2 \cdot 3) \cdot (2 \cdot 5)$ $\quad 2 \cdot 3$, $2 \cdot 5$ are both prime.

Also $60 = 2 \cdot (2 \cdot 15)$ $\quad 2$, $2 \cdot 15$ are both prime.

Thus the factorisation is not unique.