Math 1C03 Introduction to Mathematical Reasoning
Term 2 Winter 2014–2015
Problem Sheet 2: division and euclidean algorithms
to be completed by Monday January 19 2015

1) Assume that a, b, c are integers. Prove the following assertions.
   i) If $a|b$ and $b|c$ then $a|c$.
   ii) If $a|b$ and $a|c$ then $a|(bx + cy)$ for any integers x, y.
   iii) If $a|b$ then $|a| \leq |b|$.

2) The quotient/remainder theorem asserts that, for any integers a, b, there exist unique integers q, r such that $a = bq + r$ and $0 \leq r < |b|$.
   i) Give an example to show that q and r are not unique if the restriction on the size of r is omitted.
   ii) Find q and r for the following pairs of numbers
      (i) $a = 2342$, $b = 55$
      (ii) $a = -2342$, $b = 55$
      (iii) $a = 2342$, $b = -55$

3) The goal of this problem is to fill in the details of the proof of the GCD characterization theorem, 2.24 in the text.
   (a) State the assumptions of the theorem.
   (b) State the conclusion of the theorem.
   (c) Why is d not allowed to be negative?
   (d) Prove the theorem in the case $d = 0$.
   (e) What two properties have to be shown about d in order to deduce the conclusion of the theorem.
   (f) Prove the theorem in the case $d > 0$.

4) For the following pairs of numbers a, b use the (extended) euclidean algorithm to find $\gcd(a, b)$ and then find x and y so that $\gcd(a, b) = ax + by$.
   i) $a = 21$, $b = 15$
   ii) $a = 200$, $b = -45$
   iii) $a = 52804$, $b = 3600$

5) i) Define what it means to be *relatively prime*.
   ii) Suppose that a and b are relatively prime and that b and c are relatively prime. Prove or give a counterexample to the assertion that a and c are also relatively prime.
   iii) Suppose that a and c are relatively prime and that b and c are relatively prime. Prove or give a counterexample to the assertion that ab and c are relatively prime.
   iv) Prove that any two consecutive integers are relatively prime.