Math 1C03 Problem Sheet # 4 Solutions

1) $P(n)$ asserts $\sum_{i=1}^{n} i^3 = \frac{1}{4} n^2 (n+1)^2$.

Prove that $P(n)$ holds for all n by induction on n.

Base case $n=1$:

$$\sum_{i=1}^{1} i^3 = 1^3 = 1.$$

$$\frac{1}{4} 1^2 (1+1)^2 = \frac{1}{4} \cdot 1 \; 2^2 = 1.$$

So $P(1)$ is true.

Inductive step: assume $P(k)$ is true; that is,

$$\sum_{i=1}^{k} i^3 = \frac{1}{4} k^2 (k+1)^2.$$

Prove $P(k+1)$: $\sum_{i=1}^{k+1} i^3 = \frac{1}{4} (k+1)^2 (k+1+1)^2$

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$$

$$= \frac{1}{4} k^2 (k+1)^2 + (k+1)^3 \quad \text{by I H}$$

$$= (k+1)^2 \left[\frac{1}{4} k^2 + k + 1 \right]$$

$$= \frac{1}{4} (k+1)^2 (k^2 + 4k + 4)$$

$$= \frac{1}{4} (k+1)^2 (k+2)^2.$$ Thus $P(k+1)$ holds, if $P(k)$ holds.

2) i) $\sum_{i=1}^{1} i\cdot i! = 1\cdot 1! = 1$

$$\sum_{i=1}^{2} i\cdot i! = 1\cdot 1! + 2\cdot 2! = 1 + 4 = 5 = 3! - 1$$

$$\sum_{i=1}^{3} i\cdot i! = 1\cdot 1! + 2\cdot 2! + 3\cdot 3! = 5 + 3\cdot 6 = 23 = 4! - 1$$

$$\sum_{i=1}^{4} i\cdot i! = 23 + 4\cdot 4! = 23 + 4\cdot 24 = 119 = 5! - 1$$

$$\sum_{i=1}^{5} i\cdot i! = 119 + 5\cdot 5! = 119 + 5\cdot 120 = 719 = 6! - 1$$

ii) Guess: $\sum_{i=1}^{n} i\cdot i! = (n+1)! - 1$

iii) Prove by induction on n that
$$\sum_{i=1}^{n} i\cdot i! = (n+1)! - 1.$$

Base case given by i) above.

Inductive step: assume that $\sum_{i=1}^{k} i\cdot i! = (k+1)! - 1$.

Prove that $\sum_{i=1}^{k+1} i\cdot i! = (k+1+1)! - 1$

$$\sum_{i=1}^{k+1} i\cdot i! = \sum_{i=1}^{k} i\cdot i! + (k+1)(k+1)!$$

$$= (k+1)! - 1 + (k+1)(k+1)! \quad \text{by I H}$$

$$\cancel{= (k+1)!\,[}$$

$$= (k+1)!\left[1 + k+1\right] - 1$$

$$= (k+1)!\,(k+2) - 1$$

$$= (k+2)! - 1.$$

Thus the statement holds for $k+1$, assuming that it holds for k.

Hence, by induction, the statement holds for all n.

3) $P(n)\ :\ 5 \mid (8^n - 3^n)$.

Base case $n=1$: $8^1 - 3^1 = 5$, so is divisible by 5.
thus $P(1)$ holds.

Inductive step: Assume ~~the~~ $P(k)$ holds; that is, $5 \mid (8^k - 3^k)$, ~~th~~ so there is an integer q such that $8^k - 3^k = 5q$.

Prove $P(k+1)$.

$$8^{k+1} - 3^{k+1} = 8^k 8 - 3^k 3$$

$$= (5q + 3^k)8 - 3^k 3$$

$$= 5\cdot 8\cdot q + 3^k\cdot 8 - 3^k 3$$

$$= 5 \cdot 8 \cdot q + 3^k (8-3)$$
$$= 5 \cdot 8q + 3^k \cdot 5.$$

Thus $8^{k+1} - 3^{k+1}$ is divisible by 5.

Hence $P(k) \Rightarrow P(k+1)$.

By induction, $P(n)$ is true for all n.

4) $P(n)$: $3^{2n} \equiv 1 \mod 8$.

Prove $P(n)$ is true for all n by induction on n.

Base case $n = 1$: $3^2 = 9$ ~~the~~
$$\equiv 1 \mod 8.$$
So $P(1)$ holds.

Inductive step: Assume $P(k)$; that is $3^{2k} \equiv 1 \mod 8$, so $3^{2k} - 1 = 8q$ for some integer q.

$$3^{2(k+1)} = 3^{2k} \cdot 3^2$$
$$= (8q+1)\,9 \qquad \text{by IH}$$
$$= 8q + 8 + 1$$

$$= 8(q+1)+1$$

So $3^{2(k+1)} - 1 = 8(q+1)$.

Thus $3^{2(k+1)} \equiv 1 \mod 8$, i.e. $P(k+1)$ is true.

So $P(k) \implies P(k+1)$.

Hence, by induction, $P(n)$ is true for all n.

5) (i). Let $P(n)$ be the statement $n^p \equiv n \mod p$, where p is a fixed prime number. Prove $P(n)$ holds for all positive integers n by induction on n.

Base case $n=1$: $1^p = 1 \equiv 1 \mod p$. So $P(1)$ holds.

Inductive step: assume $P(k)$ holds, so

$k^p \equiv k \mod p$, or $k^p - k = pq$ for some $q \in \mathbb{Z}$.

$$(k+1)^p = k^p + \binom{p}{1}k^{p-1} + \binom{p}{2}k^{p-2} + \dots + \binom{p}{p-1}k + 1$$

by the binomial theorem

$$= k^p + pk^{p-1} + \frac{p(p-1)}{2!}k^{p-2} + \dots + pk + 1$$

$$= k^p + 1 + p\left(k^{p-1} + \frac{p-1}{2!}k^{p-2} + \dots + k\right)$$

Note: need to know $\binom{p}{k}$ is an integer 4-33 and is divisible by p if $k \neq 0, p$. Can you prove this? Needs p prime.

Thus $(k+1)^p \equiv k^p + 1 \mod p$

$\equiv k+1 \mod p$ by the induction hypothesis.

Thus $P(k) \Rightarrow P(k+1)$.

Hence $P(n)$ is true for all n by induction.

ii) We have $n^p \equiv n \mod p$, that is,

$n^p - n = pq$ for some integer q.

Then $n(n^{p-1} - 1) = pq$.

Since $p \mid n(n^{p-1}-1)$, we can conclude that $p \mid (n^{p-1}-1)$, provided $p \nmid n$ (as p is prime).

Thus $p \mid (n^{p-1}-1)$ for all n s.t. $p \nmid n$.

6) $S_n = \{a_1, a_2, \ldots, a_n\}$.

i) S_1 has subsets ϕ, $\{a_1\}$. 2

S_2 has subsets ϕ, $\{a_1\}$, $\{a_2\}$, ~~$\{a_2\}$~~

$\{a_1, a_2\}$ 4

S_3 has subsets ϕ

$\{a_1\}$ $\{a_2\}$ $\{a_3\}$

$\{a_1, a_2\}$ $\{a_1 a_3\}$ $\{a_2 a_3\}$

$\{a_1, a_2, a_3\}$ 8

S_4 has subsets ϕ

$\{a_1\}$ $\{a_2\}$ $\{a_3\}$ $\{a_4\}$

$\{a_1 a_2\}$ $\{a_1 a_3\}$ $\{a_1 a_4\}$

$\{a_2 a_3\}$ $\{a_2 a_4\}$ $\{a_3 a_4\}$

$\{a_1 a_2 a_3\}$ $\{a_1 a_2 a_4\}$

$\{a_1 a_3 a_4\}$ $\{a_2 a_3 a_4\}$

$\{a_1 a_2 a_3 a_4\}$ 16.

iii) List all subsets of S_4 - 16.
For each one, add a_5 to the subset - 16
gives 32 altogether

List all subsets of S_n - 2^n sets.
To each set, add a_{n+1} - another 2^n sets.
Altogether, $2^{n+1} = 2(2^n)$ subsets.

iii) $P(n)$ every set with n elements has 2^n subsets.
Prove for all n by induction on n.
$P(1)$: Any set with one element has itself and the empty set as subsets; so $2^1 = 2$ subsets.

$P(k) \Rightarrow P(k+1)$.

Let A be a subset of $S_{k+1} = S_k \cup \{a_{k+1}\}$.
Then <u>either</u> A is a subset of S_k; this can happen in 2^k ways, by IH
<u>or</u> $a_{k+1} \in A$ and $A \setminus \{a_{k+1}\}$ is a subset of S_k; this can happen in 2^k ways.
So in total there are $2 \cdot 2^k = 2^{k+1}$ subsets.

Hence, by induction, $P(n)$ holds for all n.