Math 1C03 Winter 2014-15 Problem Sheet 6

1) $\mathbb{Q} = \{ [(a,b)] : a \in \mathbb{Z}, b \in \mathbb{Z}^{\neq 0} \}$

where $[(a,b)] = \{ (x,y) \in \mathbb{Z} \times \mathbb{Z}^{\neq 0} : ay = bx \}$.

i) Show that multiplication is well-defined; that is, if $[(a,b)] = [(a',b')]$ and $[(c,d)] = [(c',d')]$ then $[(ac,bd)] = [(a'c', b'd')]$.

Given the assumption, we know that

$$ab' = a'b \quad \text{and} \quad cd' = c'd. \qquad (*)$$

We want to show that $acb'd' = a'c'bd$.

Well,
$$\begin{aligned} acb'd' &= ab'cd' \quad \text{(rearranging)} \\ &= a'bc'd \quad \text{by } * \\ &= a'c'bd \quad \text{(rearranging)}, \end{aligned}$$

as required.

ii) Consider $f : \mathbb{Q} \to \mathbb{Z}$

$$f([a,b]) = a+b.$$

$[(1,2)] = [(2,4)]$, as $1 \cdot 4 = 4 = 2 \cdot 2$.

$$\left.\begin{aligned} \text{But } f([(1,2)]) &= 1+2 = 3 \\ f([(2,4)]) &= 2+4 = 6 \end{aligned}\right\} \text{ Thus } f([(1,2)]) \neq f([(2,4)]) \text{ so } f \text{ is not well-defined.}$$

2) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = ax+b$.

f injective: suppose $f(x) = f(y)$.

then $ax+b = ay+b$

$$ax = ay$$

$$x = y \quad (\text{as } a \neq 0).$$

thus f is injective.

f surjective: let $y \in \mathbb{R}$. Find $x \in \mathbb{R}$

so that $f(x) = y$

i.e. $ax+b = y$

$$ax = y-b$$

$$x = \frac{y-b}{a} \quad (\text{as } a \neq 0).$$

then $f\left(\frac{y-b}{a}\right) = a\left(\frac{y-b}{a}\right)+b = y-b+b = y$.

As x exists for any $y \in \mathbb{R}$, f is surjective.

3) see video (notes attached)

3) $f: \mathbb{Z} \to \mathbb{Z}$, $f(n) = \begin{cases} n+5, & \text{if } n \text{ is even} \\ n-5, & \text{if } n \text{ is odd} \end{cases}$

$f(0) = 0 + 5$, as 0 is even
$= 5$

$f(12) = 12 + 5$, as 12 is even
$= 17$

$f(31) = 31 - 5$, as 31 is odd
$= 26$

$f(-5) = -5 - 5$, as -5 is odd
$= -10$

$f(-12) = -12 + 5$, as -12 is even
$= -7$

Conjecture: if n is even, $f(n)$ is odd
if n is odd, $f(n)$ is even.

Yes: because, if n is odd,
$f(n) = n - 5$ and odd - odd = even
if n is even, $f(n) = n + 5$
and even + odd = odd

Show f injective.

$f(m) = f(n) \Rightarrow n = m$.

But what is $f(m)$? Depends on m.

Show contrapositive: $n \neq m \Rightarrow f(m) \neq f(n)$.

Assume $n \neq m$.

Case 1 n, m both even and $n \neq m$.

Then $f(m) = m+5$
$f(n) = n+5$

$$f(m) = f(n) \Rightarrow m+5 = n+5$$
$$m = n.$$

Case 2 n, m both odd, $n \neq m$.

Then $f(m) = m-5$
$f(n) = n-5$

$$f(m) = f(n) \Rightarrow m-5 = n-5$$
$$m = n \quad \text{※}.$$

Case 3 n even, m odd, $n \neq m$.

Then $f(m) = m+5$ $\quad f(m) = f(n)$
$f(n) = n-5$ $\quad \Rightarrow m+5 = n-5$
$m-n = 10$

but $m-n$ is odd ※.

Show f surjective:

for any $x \in \mathbb{Z}$ there is $n \in \mathbb{Z}$ s.t.

$$f(n) = x.$$

<u>Case 1</u> x is even. x must come from a n odd number n.

$f(n) = n - 5$ as n odd

If $n - 5 = 2y$

then $n = 2y + 5$.

Check: $f(n) = f(2y+5) = 2y + 5 - 5$
$= 2y$
$= x$.

<u>Case 2</u> x is odd, $x = 2y+1$. x must come from an even number n.

$f(n) = n + 5$ as n even

If $n + 5 = 2y + 1$

then $n = 2y - 4$.

Check: $f(n) = f(2y-4) = 2y - 4 + 5$
$= 2y + 1$
$= x$.

Find f^{-1} explicitly.

If $x = 2y$ then $x = f(2y+5)$.

If $x = 2y+1$ then $x = f(2y-4)$.

So $$f^{-1}(x) = \begin{cases} x+5, & \text{if } x \text{ is even} \\ x-5, & \text{if } x \text{ is odd} \end{cases}$$

Check:

if x even $$\begin{aligned} f(f^{-1}(x)) &= f(x+5) \\ &= x+5-5 \quad \text{as } x+5 \text{ odd} \\ &= x. \end{aligned}$$

if x odd $$\begin{aligned} f(f^{-1}(x)) &= f(x-5) \\ &= x-5+5 \quad \text{as } x-5 \text{ even} \\ &= x. \end{aligned}$$