Math 1C03 Winter 2014-15 Problem Sheet 7 Solutions

1) i) Prove that there is no injective function from a set with $n+1$ elements to a set with n elements, for any n.

Base case: $n=1$. Let $A = \{a_1, a_2\}$ be any set with 2 elements and $B = \{b_1\}$ be any set with one element. Let $f: A \to B$ be any function. As f is defined on all of A, $f(a_1) \in B$, so $f(a_1) = b_1$. Then $f(a_2) \in B$, so also $f(a_2) = b_1$. But $a_1 \neq a_2$, so f is not injective.

Induction ~~step~~ hypothesis: assume that for any set of size $k+1$ there is no injective function to a set of size k.

$\underline{k+1}$: Let $A = \{a_1 \dots a_{k+2}\}$ be any set of size $k+2$. Let $B = \{b_1 \dots b_{k+1}\}$ be any set of size $k+1$. Let $f: A \to B$ be any function. Let $A' = \{a_1 \dots a_{k+1}\} = A \setminus \{a_{k+2}\}$, and consider $f|_{A'}$; the restriction of f to A'.

If $f|_{A'}$ is not injective, then also f is not injective and there is nothing to show. So assume $f|_{A'}$ is injective. Then $f|_{A'}(A') \subseteq B$ and by the inductive hypothesis, $f\restriction_{A'}(A')$ ~~does not have k elements~~ has at least $k+1$ elements. So $f\restriction_{A'}(A') = B$. But $f(a_{k+2}) \in B$, so $f(a_{k+2}) = f(a_i)$ for some $i < k+2$. Thus f is not injective.

ii) Suppose A has $n+1$ elements, B has n elements and there is a function $g: B \to A$ which is surjective.

Define $f: A \to B$ by

$$f(a_i) = b_j, \text{ where } j \text{ is the least index such that } g(b_j) = a_i.$$

As g is surjective, f is defined for all elements of A. Suppose $f(a_i) = f(a_k)$. Then

$g(f(a_i)) = g(f(a_k))$, as g is a function,

so $a_i = a_j$. Thus f is injective.

But this is a contradiction with i).

2) $f: \mathbb{N} \to \mathbb{E}$ $\quad f(n) = 2n$

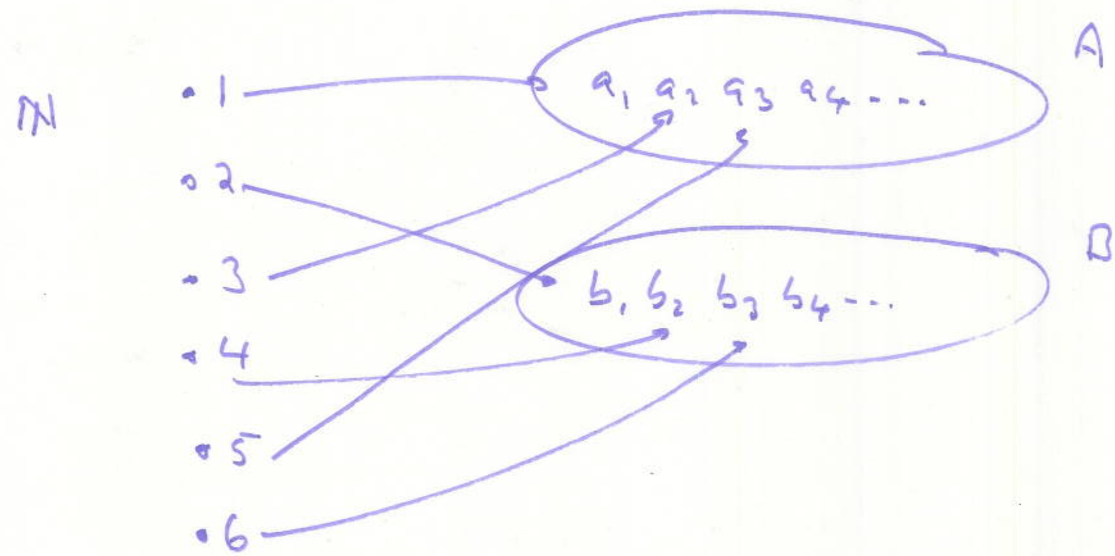
$g: \mathbb{N} \to \mathbb{O}$ $\quad g(n) = 2n-1$

$h: \mathbb{N} \to \mathbb{Z}$ $\quad h(n) = \begin{cases} \frac{n}{2}, & n \text{ even} \\ -\frac{(n+1)}{2}, & n \text{ odd} \end{cases}$

3) $f : \mathbb{N} \to A$
$g : \mathbb{N} \to B$ bijections

Define a bijection $h : \mathbb{N} \to A \cup B$.

If $A \cap B = \phi$, proceed as follows



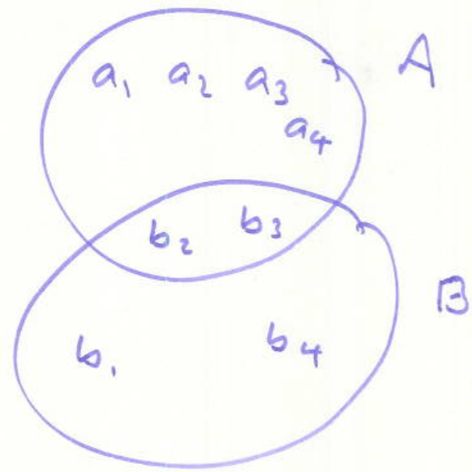
~~$h(n) = f(\frac{n}{2})$ if n is~~

$$h(n) \begin{cases} \text{uses } f & \text{if } n \text{ is odd} \\ \text{uses } g & \text{if } n \text{ is even} \end{cases}$$

so $$h(n) = \begin{cases} f\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd} \\ g\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \end{cases}$$

and h is injective and surjective because f and g both are.

$A \cap B \neq \emptyset$?



$h(1) = a_1$

$h(2) = b_1$

$h(3) = a_2$

$h(4) = \cancel{b_2}\ \cancel{b_3}\ b_4$

$h(5) = a_3$

$h(6) = \cancel{b_3}\ \cancel{b_4}\ b_5$

$h(7) = a_4$

$\vdots$

$$h(n) = \begin{cases} f\left(\frac{n+1}{2}\right), & \text{if } n \text{ is odd} \\ g(m), & \text{if } n \text{ is even, where } m \text{ is the least integer} \geq \frac{n}{2} \\ & \text{such that } g(m) \notin A \text{ and } g(m) \neq h(n') \text{ for any } n' < n. \end{cases}$$

If $A_1 \ldots A_n$ are countably infinite sets, then so is $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$.

Proof by induction on n.

Base case: $n=1$. A_1 is countably infinite by assumption.

IH $A_1 \cup \cdots \cup A_k$ is countably infinite.

Inductive step $k+1$: $\bigcup_{i=1}^{k+1} A_i = (\bigcup_{i=1}^{k} A_i) \cup A_{k+1}$.

By IH, there is bijection $f: \mathbb{N} \to \bigcup_{i=1}^{k} A_i$.

By assumption, there is bijection $g: \mathbb{N} \to A_{k+1}$.

Put these together as in part c):

Define $h: \mathbb{N} \to \bigcup_{i=1}^{k} A_i \cup A_{k+1}$ by

$$h(n) = \begin{cases} f(\frac{n+1}{2}), & \text{if } n \text{ is odd} \\ g(m), & \text{if } n \text{ is even, where } m \text{ is least integer st.} \\ & g(m) \notin \bigcup_{i=1}^{k} A_i \text{ and} \\ & g(m) \neq h(n') \text{ for any } n' < n. \end{cases}$$

4) $P(A) = \{X : X \subseteq A\}$.

Assume A is countably infinite. Prove that there is no surjective function from A to $P(A)$.

So consider any function $f : A \to P(A)$. We find a subset Y of A which is not in the range of f. It follows that f is not surjective. As this construction will work for any f, it follows that no function from A to $P(A)$ is surjective.

~~Let~~ We define Y by giving the condition on any element of Y for it to be in Y.

For any $a \in A$, if $a \in f(a)$ then $a \notin Y$.
if $a \notin f(a)$ then $a \in Y$.

Thus $Y = \{a \in A : a \notin f(a)\}$.

By definition, for any $a \in A$, $f(a) \neq Y$.

Because if $f(a) = Y$ then if $a \in Y$ then $a \notin f(a)$
so $a \notin Y$.

And if $a \notin Y$ then $a \notin f(a)$ so $a \in Y$.

Either possibility leads to a contradiction.