Math 1C03 Introduction to Mathematical Reasoning
Term 2 Winter 2014–2015
Problem Sheet 7: cardinality
to be completed by Monday March 9 2015

1) The Pigeonhole Principle states that there is no injective function from a set with $n+1$ elements to a set with n elements (if you have $n+1$ pigeons and n pigeonholes to put them into, at least one of the pigeonholes ends up with more than one pigeon in it).
   i) Prove the principle by induction on n.
   ii) Hence prove that there is no surjective function from a set with n elements to a set with $n+1$ elements.

2) The following sets are all countably infinite: $\mathbb{E}$ (the set of even integers), $\mathbb{O}$ (the set of odd integers) and $\mathbb{Z}$ (the set of all integers). Find explicit bijections between $\mathbb{N}$ and each of $\mathbb{E}$, $\mathbb{O}$ and $\mathbb{Z}$, and verify that the maps are bijections.

3) Let A and B be two countably infinite sets, with bijections $f : \mathbb{N} \to A$, $g : \mathbb{N} \to B$.
   a) Prove that $A \cup B$ is also countably infinite. This is a little tricky, as you cannot assume that A and B are disjoint. Even if you cannot write down the function explicitly, describe how it should work.
   b) Generalize to prove that for any $n \geq 2$, the union of n countably infinite sets is countably infinite. This is most easily done by induction on n; a) does the base case.

4) For any set A, we define the *power set* of A, $\mathcal{P}(A)$, to be the set of all subsets of A:
$$\mathcal{P}(A) = \{X : X \subseteq A\}.$$
In a previous Problem Sheet, you proved that if A has n elements then $\mathcal{P}(A)$ has 2^n elements. It follows from the Pigeonhole Principle that there is no surjective function from A to $\mathcal{P}(A)$ if A is a finite set. Now suppose that A is a countably infinite set. Use the idea of Cantor diagonalization to prove that there is no surjective function from A to $\mathcal{P}(A)$. We define the quantity $2^{|A|}$ to be the cardinality of $\mathcal{P}(A)$.