Math 1C03 Winter 2014-15 Problem Sheet 8 Solutions

2) $z = r\,\mathrm{cis}(\theta)$, $w = s\,\mathrm{cis}(\varphi)$

(a) $zw = r\,\mathrm{cis}(\theta)\, s\,\mathrm{cis}(\varphi) = rs\,\mathrm{cis}(\theta+\varphi)$.

So $|zw| = rs = |z||w|$.

Or $z = x+iy$, $w = u+iv$

$$zw = (x+iy)(u+iv) = xu - yv + i(yu+xv)$$

$$|zw| = \sqrt{(xu-yv)^2 + (yu+xv)^2}$$

$$= \sqrt{x^2u^2 - \cancel{2xuyv} + y^2v^2 + y^2u^2 + \cancel{2yuxv} + x^2v^2}$$

$$= \sqrt{x^2u^2 + y^2v^2 + y^2u^2 + x^2v^2} \qquad (*)$$

$$|z||w| = \sqrt{x^2+y^2}\sqrt{u^2+v^2}$$

$$= \sqrt{x^2u^2 + x^2v^2 + y^2u^2 + y^2v^2} \qquad (**)$$

As $(*) = (**)$, $|zw| = |z||w|$.

(b) Take $w = -z$. Then $|z+w| = |z-z| = |0| = 0$.

$|z| + |w| = |z| + |-z| = 2|z| \neq 0$

if $z \neq 0$.

c) Let $S = \{ z \in \mathbb{C} : |z+2| = |z| + 2 \}$

$|z+2| = |x+2+iy| = \sqrt{(x+2)^2+y^2}$

$|z|+|2| = \sqrt{x^2+y^2} + 2$

For any $z \in S$, $\sqrt{(x+2)^2+y^2} = \sqrt{x^2+y^2} + 2$

squaring: $(x+2)^2+y^2 = x^2+y^2+4+4\sqrt{x^2+y^2}$

$4x \cancel{+4} = 4\sqrt{x^2+y^2}$, so $x \geq 0$

square again: $x^2 \cancel{(x+1)^2} = x^2+y^2$, so $y = 0$.

Thus $S = \{ z = x+iy : y = 0, x \geq 0 \}$ = real half line.

d) $S = \{ z \in \mathbb{C} : |z+3i| = |z| + |3i| \}$.

$|z+3i| = |x+i(y+3)| = \sqrt{x^2+(y+3)^2}$

$|z|+|3i| = \sqrt{x^2+y^2} + 3$

For any $z \in S$, $\sqrt{x^2+(y+3)^2} = \sqrt{x^2+y^2} + 3$

$x^2+(y+3)^2 = x^2+y^2+9+6\sqrt{x^2+y^2}$

$6y = 6\sqrt{x^2+y^2}$ so $y \geq 0$

$y^2 = (x^2+y^2)$ so $x = 0$.

$S = \{ z = x+iy : x = 0, y \geq 0 \}$ = imaginary half line.

e) $S = \{ z \in \mathbb{C} : |z + 2+3i| = |z| + |2+3i| \}$

$|z + 2+3i| = \sqrt{(x+2)^2 + (y+3)^2}$

$|z| + |2+3i| = \sqrt{x^2+y^2} + \sqrt{13}$

For all $z \in S$, $\sqrt{(x+2)^2+(y+3)^2} = \sqrt{x^2+y^2} + \sqrt{13}$

$(x+2)^2 + (y+3)^2 = x^2+y^2+13 + 2\sqrt{13}\sqrt{x^2+y^2}$

$4x + 6y = 2\sqrt{13}\sqrt{x^2+y^2}$, $2x+3y \geq 0$

$4x^2+9y^2+12xy = 13(x^2+y^2)$

$9x^2+4y^2-12xy = 0$

$(3x-2y)^2 = 0$

$3x = 2y$

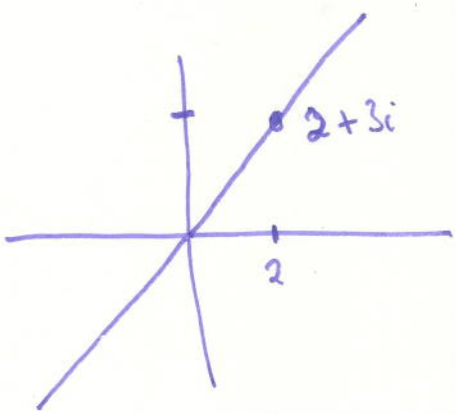
$y = \frac{3}{2}x$ and $\cancel{y \geq -\frac{2}{3}x}$

$2x+3y \geq 0$

$2x + 3\frac{3}{2}x \geq 0$

$13x \geq 0$

$x \geq 0$



S is the half-line through the point $2+3i$.

f) $S = \{ z \in \mathbb{C} : |z + (u+iv)| = |z| + |u+iv| \}$

The same calculation will give that S is the half-line through $u+iv$.

3) a) $S = \{ z \in \mathbb{C} : |z| = 10 \}$.

Writing $z = r(\cos\theta + i\sin\theta)$, if $z \in S$ then $r = 10$.
So S is the set of complex numbers with distance 10 to the origin and any argument; that is, S is a circle centre the origin radius 10.

b) $S = \{ z \in \mathbb{C} : |z-4| = 10 \}$

S is the set of points whose distance from the point 4 is equal to 10. So S is a circle centre 4, radius 10.

If $|z-4| = \sqrt{(x-4)^2 + y^2} = 10$

then $(x-4)^2 + y^2 = 100$ - this is the equation of a circle, centre $(4,0)$ radius 10.

c) $S = \{ z \in \mathbb{C} : |z - (4+3i)| = 10 \}$

This must be a circle centre $(4+3i)$ radius 10

d) $\{ z \in \mathbb{C} : |z - (u+iv)| = r \}$ is a circle centre (u,v) radius r.

4) $f(z) = \dfrac{az+b}{cz+d}$, $ab-bc \neq 0$.

a) f is defined provided $cz+d \neq 0$ ie $z \neq -d/c$.

Suppose $f(z) = f(w)$. Then

$$\frac{az+b}{cz+d} = \frac{aw+b}{cw+d}$$

$$(az+b)(cw+d) = (aw+b)(cz+d)$$

$$\cancel{aczw} + adz + bcw + \cancel{bd} = \cancel{acwz} + bcz + adw + \cancel{bd}$$

$$z(ad-bc) = w(ad-bc)$$

$$(z-w)(ad-bc) = 0$$

as $ab-bc \neq 0$, $z-w=0$ ie. $z=w$.

b) Suppose $w \in \operatorname{ran}(f)$. Then there is z s.t.

$\dfrac{az+b}{cz+d} = w$. So z satisfies:

$$az+b = w(cz+d)$$

$$z(a-wc) = wd-b$$

$$z = \frac{dw-b}{cw-a}.$$

Such a z exists provided $w \neq a/c$. Thus

$\operatorname{ran}(f) = \{ w : w \neq a/c \}$.

c) $f(z) = z + b/d = z + e$.

This is a translation; every point is moved by e.

The circle $\{z \in \mathbb{C} : |z-(u+vi)| = r\}$ must move to a circle with center $u+vi+e$ and radius r.

Check: $|f(z) - (u+iv+e)| = |z+e-(u+iv+e)|$
$= |z-(u+iv)| = r$.

5) Write w_1, w_2, w_3, w_4, w_5 for the five 5th roots of unity. These are

$w_1 = e^{2\pi/5\, i}$, $w_2 = e^{\frac{4\pi}{5} i}$, $w_3 = e^{\frac{6\pi}{5} i}$, $w_4 = e^{\frac{8\pi}{5} i}$,
$w_5 = 1$.

Notice that $w_2 = w_1^2$, $w_3 = w_1^3$, $w_4 = w_1^4$.

Now w_1 satisfies $w_1^5 - 1 = 0$. But also,

$$w_1^5 - 1 = (w_1 - 1)(w_1^4 + w_1^3 + w_1^2 + w_1 + 1)$$

as $w_1 \neq 1$, $w_1^4 + w_1^3 + w_1^2 + w_1 + 1 = 0$

That is, $w_4 + w_3 + w_2 + w_1 + w_5 = 0$, as required.

The expansion $w^n - 1 = (w-1)(w^{n-1} + \dots + w + 1)$ for any positive integer n. And the n th roots of unity are all powers of $w_1 = e^{\frac{2\pi}{n} i}$.

6) $\int e^{(a+ib)x}dx = \frac{1}{a+ib}e^{(a+ib)x}$ if

integration with complex coefficients works the same as integration with real coefficients.

Now $e^{(a+ib)x} = e^{ax+ibx} = e^{ax}e^{ibx}$

$= e^{ax}(\cos(bx) + i\sin(bx))$

for x real.

So $\int e^{(a+ib)x}dx = \int e^{ax}\cos(bx)dx + i\int e^{ax}\sin(bx)dx$.

So $\int e^{ax}\cos(bx)dx = \mathrm{re}\left(\frac{1}{a+ib}e^{(a+ib)x}\right)$

and $\int e^{ax}\sin(bx)dx = \mathrm{im}\left(\frac{1}{a+ib}e^{(a+ib)x}\right)$.

$$\frac{1}{a+ib}e^{(a+ib)x} = \frac{a-ib}{a^2+b^2}e^{ax}(\cos(bx) + i\sin(bx)).$$

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$$= \frac{e^{ax}}{a^2+b^2}(a\cos(bx) + b\sin(bx))$$
$$+ \frac{i\,e^{ax}}{a^2+b^2}(a\sin(bx) - b\cos(bx)).$$

Thus $\int e^{ax}\cos(bx)dx = \frac{1}{a^2+b^2}e^{ax}(a\cos(bx) + b\sin(bx))$

and $\int e^{ax}\sin(bx)dx = \frac{1}{a^2+b^2}e^{ax}(a\sin(bx) - b\cos(bx))$