McMaster University Math 1XX3 Winter 2013
Midterm 2 delayed version
March 18 2013
Duration: 60 minutes

Instructor: Dr. D. Haskell

Name: SOLUTIONS

Student ID Number: ________________

This test paper is printed on both sides of the page. There are 5 question on 6 pages. You are responsible for ensuring that your copy of this test is complete. Bring any discrepancies to the attention of the invigilator. Paper for rough work is available from the invigilator.

Instructions

(1) Only the standard McMaster calculator is allowed.
(2) Additional scratch paper is available for rough work; ask the invigilator.
(3) All answers must be written in the space following the question. If you write an answer on scratch paper, indicate **clearly** where to find your answer.

Problem	Points
1 [15]	
2 [5]	
3 [6]	
4 [8]	
5 [6]	
Total [40]	

1) [15 points]

a) Given the differential equation $\dfrac{dy}{dx} = xy - x^2 + y$, with initial data $y(0) = 2$, use Euler's method with step size $h = 1$ to approximate $y(1)$.

$x_1 = 0, \; y_1 = 2$

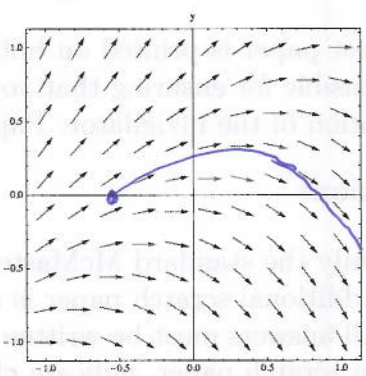
$x_2 = 1, \; y_2 = F(x_1, y_1)\, h + y_1$

$= (0 \cdot 2 - 0^2 + 2) \cdot 1 + 2$

$= 2 + 2$

$= 4$

b) On the slope field given, sketch the solution which satisfies $y(-1/2) = 0$.



c) Describe the path of a particle (both shape of curve and how the curve is traversed) which moves so that its position in the plane at any time t is given by the parametric equations $x = \sin(t)$, $y = \cos(t)$.

$x^2 + y^2 = \sin^2(t) + \cos^2(t) = 1$ so curve is a circle, centre $(0,0)$ radius 1.

$x(0) = \sin(0) = 0, \; y(0) = \cos(0) = 1.$

$x(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1, \; y(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) = 0$

circle is traversed clockwise, starting at $(0,1)$.

d) Find the cartesian coordinates of the point whose polar coordinates are $(3, 5\pi/4)$.

$r = 3, \; \theta = \frac{5\pi}{4}$

$x = r\cos\theta = 3\cos(\frac{5\pi}{4}) = 3\left(\frac{-1}{\sqrt{2}}\right)$

$y = r\sin\theta = 3\sin(\frac{5\pi}{4}) = 3 \cdot \left(\frac{-1}{\sqrt{2}}\right)$

e) A population of bacteria grows exponentially, satisfying the differential equation $\frac{dP}{dt} = 10P$. If the initial population is 200, find the function $P(t)$.

$$P(t) = 200e^{10t}$$

f) Let $A(t)$ be the surface area of a growing tumour, and let M be the final area of the tumour. Experiments show that the rate of change of A is proportional to the porduct of $\sqrt{A}$ and the difference between M and $A(t)$. Write a differential equation for $A(t)$.

$$\frac{dA}{dt} = k\sqrt{A}(M-A).$$

2) [5 points] Solve the initial value problem $\frac{dy}{dx} = \sqrt{xy}$, $y(0) = 1$.

$$\frac{dy}{dx} = \sqrt{xy}$$

$$\int \frac{1}{\sqrt{y}}\, dy = \int \sqrt{x}\, dx$$

$$2y^{1/2} = \frac{2}{3}x^{3/2} + C$$

$$y(0)=1: \quad 2 = \frac{2}{3}\cdot 0 + C \qquad \underline{C=2}$$

$$y^{1/2} = \frac{1}{3}x^{3/2} + 1$$

$$y = \left(\frac{1}{3}x^{3/2} + 1\right)^2$$

3) [6 points] Find the general solution of the linear differential equation $x^2\frac{dy}{dx} = 3xy + 4x^5e^x$.

$$x^2\frac{dy}{dx} = 3xy + 4x^5e^x$$

$$\frac{dy}{dx} = \frac{3}{x}y + 4x^3e^x$$

$$\frac{dy}{dx} - \frac{3}{x}y = 4x^3e^x.$$

$$I(x) = e^{\int -\frac{3}{x}dx} = e^{-3\ln(x)} = e^{\ln x^{-3}} = x^{-3}.$$

$$\frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y = 4e^x$$

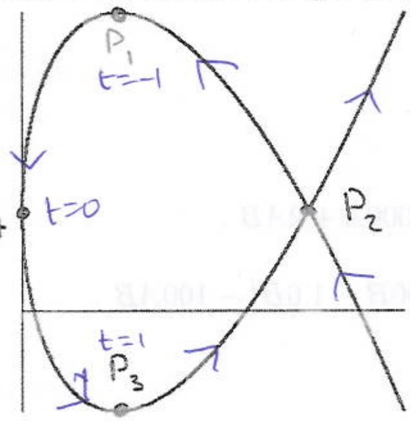
$$\int \frac{d}{dx}\left(\frac{1}{x^3}y\right)dx = \int 4e^x\,dx$$

$$\frac{1}{x^3}y = 4e^x + C$$

$$y = x^3(4e^x + C).$$

4) [8 points] The curve shown is the path of a particle which moves with parametric equations

$$x = t^2,$$
$$y = t^3 - 3t + 1.$$



a) For each point P_1, P_2, P_3, P_4 marked on the curve, find both the cartesian coordinates of the point and the value(s) t at which the point is reached.

$\underline{P_1, P_3}$ $\quad \frac{dy}{dt} = 0 \qquad \frac{dy}{dt} = 3t^2 - 3 = 3(t-1)(t+1) = 0$ when

$t=1,\ t=-1.$

$t=1 \quad y = 1-3+1 = -1. \qquad x = 1 \qquad P_3$

$t=-1 \quad y = -1+3+1 = 3 \qquad y = 1 \qquad P_1.$

$\underline{P_4}$ $\quad \frac{dx}{dt} = 0 \qquad \frac{dx}{dt} = 2t = 0$ when $t = 0$, so $x = 0$

$y = 1.$

$\underline{P_2}$ Find values of t so that $y = 1$.

$t^3 - 3t + 1 = 1$

$t^3 - 3t = 0$

$t(t^2 - 3) = 0$

$t = 0$ or $t = \pm\sqrt{3}$

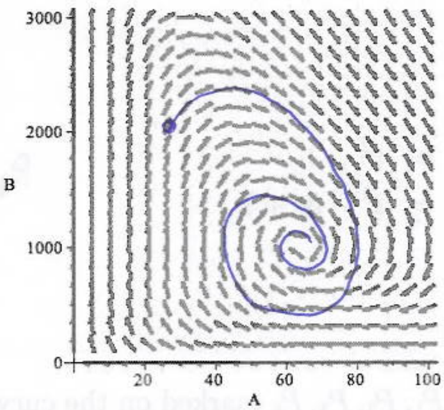
$x = 3.$

$\underline{P_2(3,1) \text{ at } t = \sqrt{3} \text{ and } t = -\sqrt{3}}$

b) Put several arrows on the curve to mark the direction in which it is traversed as t increases.

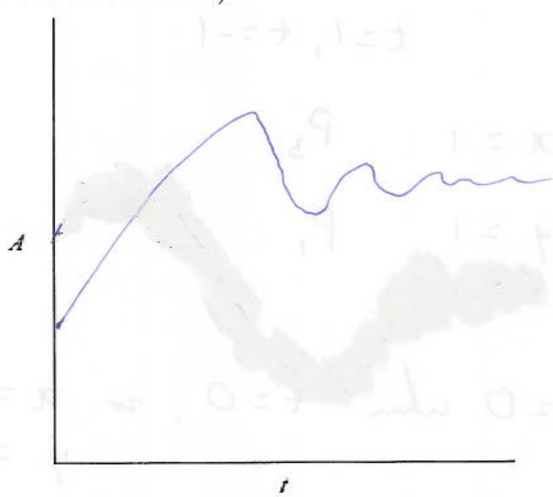
5) [6 points] The populations $A(t)$, $B(t)$ of two interacting species are modelled by the differential equations

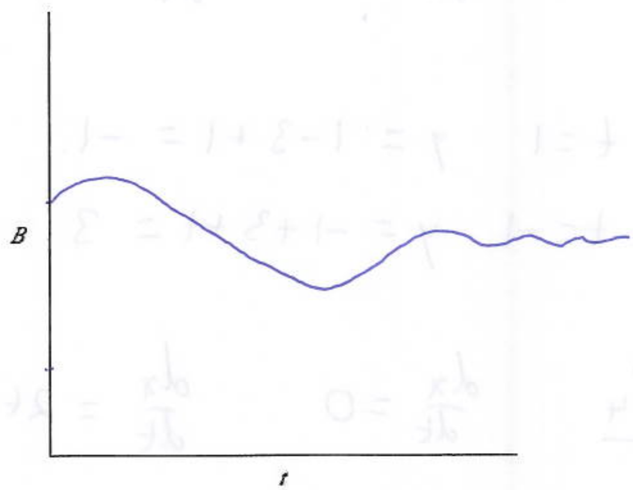
$$\frac{dA}{dt} = -2000A + 2AB\ ,$$
$$\frac{dB}{dt} = 8000B - 1.6B^2 - 100AB\ .$$



a) On the slope field given, sketch the solution curve which starts at $A = 30$, $B = 2000$ (hint: the solution approaches an equilibrium solution).

b) For the solution curve that you just drew, sketch the graphs of A against t and B against t (on the axes below).





c) Your graphs in part b) should show that $A(t)$ and $B(t)$ both approach some values. Find those values.

Eq. when $\frac{dA}{dt} = \frac{dB}{dt} = 0$.

$\frac{dB}{dt} = B(8000 - 1.6B - 100A)$
$= 0$ when $B = 0$ or

$\frac{dA}{dt} = A(-2000 + 2B) = 0$

$A = 0$ or $\underline{B = 1000}$

$8000 - 1.6B - 100A = 0$

$8000 - 1600 - 100A = 0$

$-100A = 6400$

$\underline{A = 64}$