12) [12 points]

a) What can you say about solutions to the differential equation $\frac{dy}{dx} = x^2 + y^2$ just by looking at the differential equation?

Because $x^2+y^2 \geq 0$ for all x, y, all solutions have positive derivative, so are increasing for all x, y.

b) Let C_1 be the curve defined by the equation $r = 3$, C_2 be the curve defined by the equation $x^2 + y^2 = 9$ and C_3 be the curve defined by the pair of equations $x = 3\sin(2t), y = 3\cos(\cancel{3}t)$.
2
Compare C_1, C_2, C_3.

$C_1 : r = 3$ circle centre $(0,0)$ radius 3.

$C_2 : x^2+y^2=9$ circle centre $(0,0)$ radius 3

$C_3 : \quad x = 3\sin(2t) \,, y = 3\cos(2t)$

$x^2+y^2 = 9\sin^2(2t) + 9\cos^2(2t)$
$= 9.$

The curves are all the same. C_3 is traversed repeatedly.

c) Suppose $\sum a_n$ is a convergent sum of positive terms. Does it follow that $\sum(-1)^n a_n$ converges?

As $\sum a_n$ is convergent, $a_n \to 0$.

By the alternating series test, $\sum (-1)^n a_n$ also converges.

10) [8 points] Find the improper integral $\int_1^{\infty} (1 - \frac{1}{x}) e^{\ln(x) - x}\, dx$.

$$\int \left(1 - \frac{1}{x}\right) e^{\ln(x) - x}\, dx = \int e^u \, (-du)$$

$u = \ln(x) - x$ $\qquad = -e^u + C$

$du = \left(\frac{1}{x} - 1\right) dx$ $\qquad = -e^{\ln(x) - x} + C.$

$$\int_1^{\infty} \left(1 - \frac{1}{x}\right) e^{\ln(x) - x}\, dx = \lim_{t \to \infty} \left[-e^{\ln(x) - x} \right]_1^t$$

$$= \lim_{t \to \infty} \left(-e^{\ln(t) - t} + e^{\ln(1) - 1} \right)$$

$$\lim_{t \to \infty} (\ln(t) - t) = \lim_{t \to \infty} \ln(t) \left(1 - \frac{t}{\ln(t)}\right)$$

$$= \infty \, (-\infty) = -\infty$$

(the L'Hopital's Rule on $\lim_{t \to \infty} \frac{t}{\ln(t)}$)

So $\lim_{t \to \infty} -e^{\ln(t) - t} = -\lim_{t \to \infty} e^{\ln(t) - t} = -e^{\lim_{t \to \infty} (\ln(t) - t)}$

$= 0.$

Thus $\int_1^{\infty} \left(1 - \frac{1}{x}\right) e^{\ln(x) - x}\, dx = 0 + e^{\ln(1) - 1} = \frac{1}{e}.$

7) [8 points] Test the series $\sum_{n=2}^{\infty} \frac{8^n}{2+9^n}$ for convergence. State any test you use, and be careful to check its hypotheses.

$$\sum_{n=2}^{\infty} \frac{8^n}{2+9^n} \cancel{= \sum_{n=2}^{\infty}}$$

$\frac{8^n}{2+9^n}$ is positive for all n.

$$2+9^n > 9^n$$

so $\frac{1}{2+9^n} < \frac{1}{9^n}$

and $\frac{8^n}{2+9^n} < \frac{8^n}{9^n} = \left(\frac{8}{9}\right)^n$.

The series $\sum \left(\frac{8}{9}\right)^n$ is a geometric series with ratio $\frac{8}{9} < 1$, hence convergent.

Our series $\sum \frac{8^n}{2+9^n}$ is less than $\sum \left(\frac{8}{9}\right)^n$, so by the Comparison Test, $\sum \frac{8^n}{2+9^n}$ is also convergent.

5) [8 points] Find the indefinite integral $\int \frac{1}{(x-3)(x-2)}\,dx$.

$$\frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

$$1 = A(x-2) + B(x-3)$$

$x=2$: $1 = A \cdot 0 + B(-1)$ $\quad \underline{B=-1}$

$x=3$: $1 = A(1) + B(0)$ $\quad \underline{A=1}$

$$\int \frac{1}{(x-3)(x-2)}\,dx = \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right)dx$$

$$= \ln|x-3| - \ln|x-2| + C$$

3) [8 points] Consider the curve given in parametric form by the equations $x = \theta - \sin(\theta)$, $y = 1 - \cos(\theta)$.

a) Complete the table of values.

θ	0	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$	3π	$7\pi/2$	4π
x	0	$\frac{\pi}{2}-1$	π	$\frac{3\pi}{2}+1$	2π	$\frac{5\pi}{2}-1$	3π	$\frac{7\pi}{2}+1$	4π
y	0	1	2	1	0	1	2	1	0

b) Find the points in the interval $0 \le \theta \le 4\pi$ at which the curve has a horizontal tangent line.

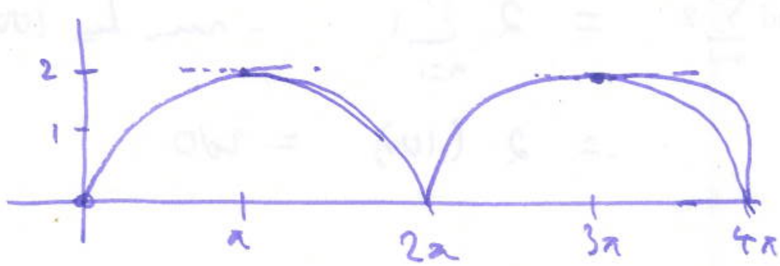
$\frac{dy}{dt} = \sin\theta = 0$ when $\theta = 0, \pi, 2\pi, 3\pi, 4\pi$.

c) Find the points in the interval $0 \le \theta \le 4\pi$ at which the curve has a vertical tangent line.

$\frac{dx}{dt} = 1 - \cos\theta = 0$ when $\cos\theta = 1$ $\theta = 0, 2\pi, 4\pi$.

at $\theta = 0, 2\pi, 4\pi$ have a singularity of some kind.

d) Sketch the curve.



1) [10 points]

a) State the Integral Test for convergence of the series $\sum_{n=0}^{\infty} a_n$.

If $a_n = f(n)$ and the function $f(x)$ is positive and decreasing then the series and the improper integral $\int_a^{\infty} f(x)\,dx$ either both converge or both diverge

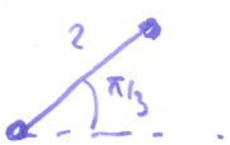
b) State the Fundamental Theorem of Calculus Part I.

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

c) Find $\int_1^2 x^2\,dx$.

$$\int_1^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_1^2 = \frac{1}{3}8 - \frac{1}{3}1 = \frac{7}{3}.$$

d) Find cartesian coordinates for the point $(2, \pi/3)$ in polar coordinates.



$$x = r\cos\theta = 2\cos\left(\frac{\pi}{3}\right) = 2\,\frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\left(\frac{\pi}{3}\right) = 2\,\frac{\sqrt{3}}{2} = \sqrt{3}$$

e) Sketch the slope field for the differential equation $\frac{dy}{dx} = y$.

$y = 0,\ \frac{dy}{dx} = 0$

$y > 0,\ \frac{dy}{dx} > 0$

$y < 0,\ \frac{dy}{dx} < 0$