McMaster University Math 1XX3 Winter 2013
Final Exam — PRACTICE version
Duration: 3 hours

Instructor: Dr. D. Haskell

Name: SOLUTIONS

Student ID Number: ______________

This test paper is printed on both sides of the page. There are 12 question on 12 pages. You are responsible for ensuring that your copy of this test is complete. Bring any discrepancies to the attention of the invigilator. More paper for rough work is available from the invigilator.

Instructions

(1) Only the standard McMaster calculator is allowed.
(2) All answers must be written in the space following the question. If you write an answer on scratch paper, indicate **clearly** where to find your answer.

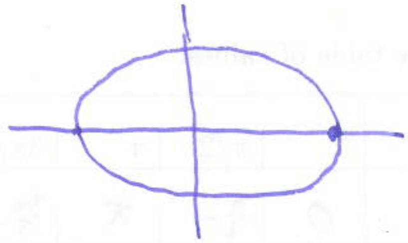
This PRACTICE version of the midterm is intended to give you an idea of the format, approximate length and approximate difficulty of the actual midterm. There is no guarantee as to the actual length and difficulty of the actual exam. In particular, the actual midterm will NOT be "just the same with the numbers changed".

2) [10 points]

a) Sketch the graph of $x^2 + 4y^2 = 1$.

$x = 0,\ y = \pm\frac{1}{2}$

$y = 0,\ x = \pm 1$



b) Give an example of a sequence which is bounded but not monotonic.

$\frac{(-1)^n}{n}$

c) Is the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent or divergent. Justify your answer briefly.

p-series with $p = 2 > 1$ so converges

d) Suppose $x(t)$, $y(t)$ are two interacting populations described by the differential equations

$$\frac{dx}{dt} = 0.01x - 0.5xy \qquad \frac{dy}{dt} = -0.01y + 0.002xy\ .$$

Which variable represents the predator population, and which is the prey? Explain briefly.

If $y = 0$, $\frac{dx}{dt} = 0.01x$ has exponential growth

If $x = 0$, $\frac{dy}{dt} = -0.01y$ has exp. decay.

x is prey, because grows in absence of y

y is predator because dies off without x.

e) Find $\sum_{n=1}^{100} 2$.

$= 2 \sum_{n=1}^{100} 1$ - sum has 100 terms

$= 2\,(100) = 200$

4) [8 points]

a) Find the Taylor series around 0 for the function $f(x) = \cos(x)$.

$f(x) = \cos(x) \quad f(0) = 1$

$f'(x) = -\sin(x) \quad f'(0) = 0$

$f''(x) = -\cos(x) \quad f''(0) = -1$

$f'''(x) = \sin(x) \quad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos(x) \quad f^{(4)}(0) = 1.$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} (-1)^n x^{2n}$$

b) Determine the number of terms of the Taylor series that are needed in order to estimate $\cos(0.1)$ with an error of less than 10^{-4}.

$$|R_m(x)| \le \frac{1}{(m+1)!} f^{(m+1)}(y)\, x^{m+1}$$

Here, $x = 0.1$, $\quad f^{(m+1)}(y) = \pm \cos(y)$ or $\pm \sin(y)$,

so $|f^{(m+1)}(y)| \le 1$ for all y.

So make $\quad \frac{1}{(m+1)!} \cdot 1 \left(\frac{1}{10}\right)^{m+1} < \frac{1}{10^4}$

take $\underline{m=3}$ $\quad \frac{1}{4!} \cdot \left(\frac{1}{10}\right)^4 < \frac{1}{10^4}$ but with $m = 2$, get $\frac{1}{3!}\left(\frac{1}{10}\right)^3 = \frac{1}{6}\frac{1}{10^3} > \frac{1}{10^4}$

c) Use your Taylor series from part a) to find $\cos(0.1)$ with an error of less than 10^{-4}.

$$f(x) \approx \underset{n=0}{1} - \underset{n=1}{\frac{1}{2!}x^2} + \underset{n=2}{\frac{1}{4!}x^4}$$ degree 3 is high enough

$$f\left(\tfrac{1}{10}\right) = 1 - \frac{1}{2!}\frac{1}{100} + \cancel{\frac{1}{24}\frac{1}{10000}} = 0.995 + \cancel{\frac{1}{240000}} =$$

6) [8 points] Consider the power series $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n(n+1)}$.

a) Find the radius of convergence.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{3^{n+1}(n+2)} \cdot \frac{3^n(n+1)}{(-1)^n x^n} \right|$$

$$= \lim_{n\to\infty} \left|\frac{x}{3}\right| \frac{(n+1)}{(n+2)}$$

$$= \frac{1}{3}|x| . 1$$

Series converges if $\frac{1}{3}|x| < 1$ i.e. $|x| < 3$.

radius of convergence is 3.

b) Find the interval of convergence.

$\underline{x=3}$ Series: $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{3^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

this is an alternating harmonic series, so converges

$\underline{x=-3}$ Series is $\sum_{n=0}^{\infty} \frac{(-1)^n(-3)^n}{3^n(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1}$

this is a harmonic series, which diverges.

Thus the interval of convergence is $(-3, 3]$.

9) [8 points] Solve the differential equation $x\ln(x)\dfrac{dy}{dx} + y = xe^x$.

$$\frac{dy}{dx} + \frac{1}{x\ln(x)}y = \frac{1}{\ln(x)}e^x$$

Fo.. linear. Find integrating factor $I(x) = e^{\int \frac{1}{x\ln(x)}dx}$

$$\int \frac{1}{x\ln(x)}dx = \int \frac{1}{u}du \qquad u = \ln(x), \quad du = \frac{1}{x}dx$$

$$= \ln(u) + C$$

$$= \ln(\ln(x)) + C.$$

$$I(x) = e^{\ln(\ln(x))} = \ln(x)$$

Multiply eqn by $I(x)$: $\ln(x)\frac{dy}{dx} + \frac{1}{x}y = e^x$.

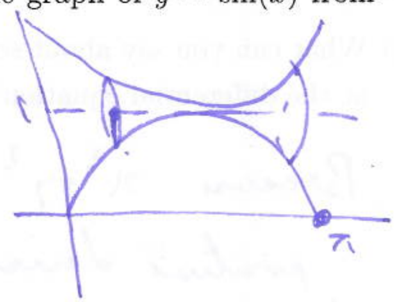
As $\frac{d}{dx}\left(\ln(x)y\right) = \ln(x)\frac{dy}{dx} + \frac{1}{x}y$,

$$\ln(x)y = \int e^x\,dx$$

$$\ln(x)y = e^x + C$$

$$y = \frac{1}{\ln(x)}\left(e^x + C\right)$$

11) [8 points] Find the volume of the solid formed by rotating the graph of $y = \sin(x)$ from $x = 0$ to $x = \pi$ around the line $y = 1$.



$$\text{volume} = \int_0^{\pi} \pi r^2 \, dx$$

cross-section is circle, radius $1 - \sin(x)$

$$\text{volume} = \int_0^{\pi} \pi (1 - \sin(x))^2 \, dx = \pi \int_0^{\pi} (1 - 2\sin(x) + \sin^2(x)) \, dx$$

$$= \pi \int_0^{\pi} \left(1 - 2\sin(x) + \frac{1}{2}(1 - \cos(2x))\right) dx$$

$$= \pi \left[x + 2\cos(x) + \frac{1}{2}x - \frac{1}{4}\sin(2x) \right]_0^{\pi}$$

$$= \pi \left[\frac{3}{2}\pi + 2\cos(\pi) - \frac{1}{4}\sin(2\pi) - \left(\frac{3}{2} \cdot 0 + 2\cos(0) - \frac{1}{4}\sin(0) \right) \right]$$

$$= \pi \left[\frac{3}{2}\pi - 2 - 0 - 0 + 2 - 0 \right]$$

$$= \frac{3}{2}\pi^2$$