Solutions to Math 3E03 Homework 6

2.116. Since D_8 and $\mathbf{Q}$ are the only nonabelian groups, they are non-isomorphic to the remaining three. From problem **2.87**, D_8 and $\mathbf{Q}$ are non-isomorphic to each other. It remains to check that $\mathbb{I}_8, \mathbb{I}_4 \times \mathbb{I}_2$ and $\mathbb{I}_2 \times \mathbb{I}_2 \times \mathbb{I}_2$ are mutually non-isomorphic. This follows from the observation that $\mathbb{I}_8$ is the only group with an element of order 8 and $\mathbb{I}_2 \times \mathbb{I}_2 \times \mathbb{I}_2$ is the only group with all non-identity elements having order 2.

2.118. Let G be a simple p-group with order p^e. Suppose that $e \geq 2$, then by Proposition **2.152**, G has a normal subgroup of order p, contradicting the assumption of G being simple. Hence we have $e = 1$ and $|G| = p$.

Conversely suppose $|G| = p$ and H is a normal subgroup of G, then by Lagrange's Theorem, H has order either 1 or p. This implies that H is either the trivial subgroup or the whole of G.

2.121. (i) By proposition 2.99, A_4 does not have subgroup of order 6. In particular, it does not have an element of order 6. On the other hand, D_{12} has an element of order 6. Hence $A_4 \ncong D_{12}$.

(ii) Recall that D_{12} is a group of order 12 with an element a of order 6 and an element b of order 2 such that $bab = a^{-1}$. Clearly $\langle a^3 \rangle$ generates a subgroup of order 2. We claim that this group is normal in D_{12}. Since D_{12} is generated by a and b, it suffices to check that aa^3a^{-1} and ba^3b^{-1} are in $\langle a^3 \rangle$. The first term is obvious and to see the same for the second term, one observes that $ba^3b^{-1} = bab^{-1}bab^{-1}bab^{-1} = a^{-1}a^{-1}a^{-1} = a^{-3} = a^3$.

Also one has $ba^2b = babbab = a^{-1}a^{-1} = (a^2)^{-1}$. Let $H = \langle a^2 \rangle \langle b \rangle$. By the product formula, this is a subset of order 6. From the relation $ba^2 = (a^2)^{-1}b$, one sees that $\langle a^2 \rangle \langle b \rangle = \langle b \rangle \langle a^2 \rangle$ and so H is a subgroup of D_{12} by problem **2.106**. Since H is a subgroup of D_{12} of index 2, it is normal in D_{12}. Now from the relation $ba^2 = (a^2)^{-1}b$, we see that H is a nonabelian group of order 6. Thus it must be isomorphic to S_3 by proposition 2.135.

We now show that $H \cap \langle a^3 \rangle = \{1\}$. Since a^3 is the only nonidentity element in $\langle a^3 \rangle$, it suffices to show that $a^3 \notin H$. But if $a^3 \in H$, then $a = a^3a^{-2} \in H$. Since b is also contained in H, it follows that $H = D_{12}$, which is a contradiction since H only has order 6. Hence we have $H \cap \langle a^3 \rangle = \{1\}$. Applying the product formula, we have $H\langle a^3 \rangle$ being a subgroup of D_{12} of order 12. Hence $D_{12} = H\langle a^3 \rangle$. Thus we have shown that $D_{12} = H \times \langle a^3 \rangle \cong S_3 \times \mathbb{I}_2$.

2.122. (i) Clearly one has $C_H(x) \subseteq H \cap C_G(x)$. On the other hand if $g \in H \cap C_G(x)$, then g is an element of H with $gx = xg$. Thus $g \in C_H(x)$.

(ii) Since H is of index 2 in G, it is a normal subgroup of G. Thus $HC_G(x)$ is a subgroup of G and it contains H. We have $2 = [G : H] = [G : HC_G(x)][HC_G(x) : H]$. Hence we have either $[G : HC_G(x)] = 1$ or $[HC_G(x) : H] = 1$. Now by the product formula, $|HC_G(x)| = \frac{|H||C_G(x)|}{|H \cap C_G(x)|} = \frac{|H||C_G(x)|}{|C_H(x)|}$. Rearranging this equation, we obtain $\frac{|C_G(x)|}{|C_H(x)|} = \frac{|HC_G(x)|}{|H|}$.

If $[G : HC_G(x)] = 1$, then $HC_G(x) = G$ and so the above formula becomes $\frac{|C_G(x)|}{|C_H(x)|} = \frac{|G|}{|H|}$. Rearranging and recalling that $|x^K||C_K(x)| = |K|$ for any group K, we obtain $|x^H| = |x^G|$.

Suppose that $[HC_G(x) : H] = 1$, then $HC_G(x) = H$ and one obtains from the above formula that $|C_G(x)| = |C_H(x)|$. Therefore we have $2|x^H||C_H(x)| = 2|H| = |G| = |x^G||C_G(x)|$ which implies that $|x^G| = 2|x^H|$.

2.127. (i) Since an element of A_n can be written as a product of an even number of transpositions, it suffices to show that a product of two transpositions can be written as product of 3-cycles. Let α, β be two transpositions.

Case 1 : $\alpha = \beta$. This is trivial.

Case 2 : $\alpha = (a\ b)$ and $\beta = (a\ c)$ where $b \neq c$. Then $\alpha\beta = (a\ c\ b)$.

Case 3 : $\alpha = (a\ b)$ and $\beta = (c\ d)$ where a, b, c, d are not equal to one another. Then $\alpha\beta = (a\ b\ c)(b\ c\ d)$.

(ii) Let $\alpha = (a\ b\ c) \in H$. By (i), it suffices to show that H contains every 3-cycles. Let β be another 3-cycle in A_n.

Case 1 : $\beta = (a\ b\ d)$ or $\beta = (a\ d\ b)$, $d \notin \{a, b, c\}$. It suffices to show that one of them lies in H since one is the inverse of the other. Since H is a normal subgroup of A_n, $(a\ b)(c\ d)(a\ b\ c)[(a\ b)(c\ d)]^{-1} = (a\ d\ b)$ is in H. (Note that we cannot use $(c\ d)$ since this is not in A_n).

Case 2 : $\beta = (a\ d\ e)$ or $\beta = (a\ e\ d)$, $d, e \notin \{a, b, c\}$. Then $(b\ d)(c\ e)(a\ b\ c)[(b\ d)(c\ e)]^{-1} = (a\ d\ e)$ is in H.

Case 3 : $\beta = (d\ e\ f)$, $d, e, f \notin \{a, b, c\}$. Then $\alpha^{-1} = (a\ c\ b)$ is in H and hence so is $(e\ f)(c\ f)(b\ e)(a\ d)(a\ c\ b)[(e\ f)(c\ f)(b\ e)(a\ d)]^{-1} = (d\ e\ f)$.

2.134. By Cauchy Theorem, G contains an element of order p. This element generates a subgroup H of order p. Let $g \in G$. Then gHg^{-1} is also a subgroup of order p. By Lagrange's Theorem, $H \cap gHg^{-1}$ has order 1 or p. Suppose this group has order 1, then by the product formula, $H(gHg^{-1})$ is a subset of G with order p^2 which is greater than mp, the order of G. This is not possible and so we must have $H \cap gHg^{-1}$ having order p. This implies that $gHg^{-1} = H \cap gHg^{-1} = H$. Hence H is a proper normal subgroup of G and so G is not simple.

2.131. Suppose there exist a subgroup H of A_5 of order 30, then H is normal in A_5 (since it is of index 2) contradicting A_5 being simple. Hence A_5 cannot have a subgroup of order 30.